Study of Enriched Contractions in Some Abstract Spaces

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of degree

Master of Science
in
Mathematics

by

Sanket Pawar
(MS23MTH005)

under the guidance of
Dr. Deepesh Kumar Patel

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur 440010 (India)
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Declaration

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This is to certify that the dissertation titled "Study of Enriched Contraction in Some Abstract Spaces" is submitted by Sanket Pawar in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics, VNIT Nagpur. The work is thorough complete and ready for final assessment.

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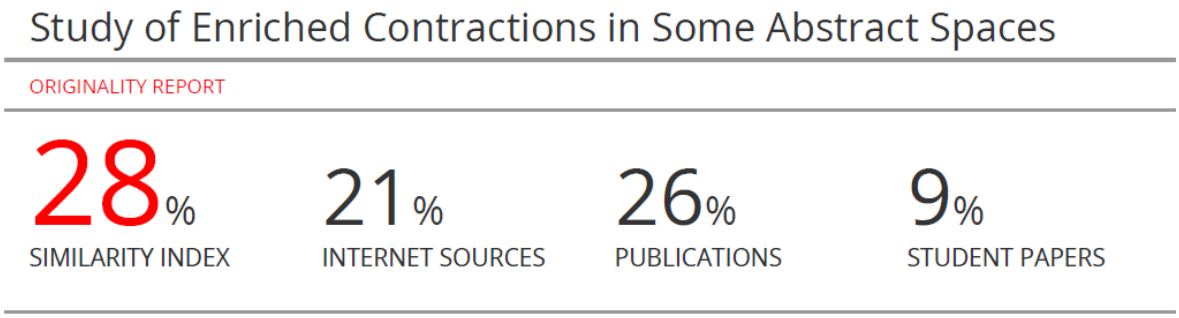

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Sanket Pawar

Abstract

In this report we study about fixed point concepts and various means to generalize it. In first Chater, we study the well known "Banach contraction principle " and further, we study large class of contractive mappings, called Enriched contraction. Later, we see the uniqueness and existence of enriched contraction in Banach spaces using Kranoselskij iterative scheme it's consequences and generalization. 2 Chapter, introduces enriched non-expansive mappings which helps in approximating fixed point of it in Hilbert spaces. Further, we see the weak convergence theorem for enriched non expansive mappings in Hilbert spaces. Then, Chapter 3 deals with more elaborative enriched contraction than previous two called Enriched Chatterjea contractions, which also uses distance between it's images to approximate fixed point of it even more precisely. Chapter 5 uses three different contraction definition and combining it provide a new one called enriched Ćirić-Reich-Rus contractions. Later, we discuss the advancement of fixed point theorems, specifically for enriched Ćirić-Reich-Rus contractions. The key progression highlighted is the extension of these theorems beyond traditional Banach spaces into more generalized settings like convex metric spaces. Furthermore, it mentions the utilization of Takahashi convex metric spaces to provide even broader generalizations of existing results in this area. And, finally we conclude it by summarizing the comprehensive analysis of these contractive mappings and their fixed point theorem's results.

This report is based on the study of following research articles:

- R. P. Agarwal, Donal O'Regan, and D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications 6, New York: Springer, (2009).
- E. Kreyszig,Introductory functional analysis with applications 17, John Wiley and Sons, (1991).

- V. Berinde, "Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces," Carpathian Journal of Mathematics 35.3, 293-304 (2019).

- V. Berinde, and Mădălina Păcurar, "Approximating fixed points of enriched contractions in Banach spaces,"Journal of Fixed Point Theory and Applications 22, 1-10 (2020).

- V. Berinde, and Mădălina Păcurar, "Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces," Journal of Fixed Point Theory and Applications 23, 1-16 (2021).

- V. Berinde, and Mădălina Păcurar, "Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces," Carpathian Journal of Mathematics 37.2, 173-184 (2021).

List of Notations

$\mathbb{C}$	: The set of cxomplex numbers
$\mathbb{R}$	: The set of real numbers
$\mathbb{N}$	: The set of natural numbers
$\rightarrow$	: maps to
$\subset$	: subset of
$S : \mathcal{X} \rightarrow \mathcal{X}$	: S is a mapping from $\mathcal{X}$ into $\mathcal{X}$
$S(\varkappa)$	: Image of $\varkappa$ under S
iff	: If and only if
max	: Maximum
sup	: supremum
min	: Minimum
inf	: Infimum
$\varnothing$	: Empty set
$\in$	: Belongs to
$\exists$	: There exist
$(\mathcal{X}, \rho)$	: A metric space with metric ρ
$\rho(\varkappa_1, \varkappa_2)$	: Distance of point $\varkappa_1$ from point $\varkappa_2$
$\forall$	: For all

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Chapter 1

Approximation of Fixed Points of Enriched Contractions in Banach Spaces

Because of its many uses in the solution of functional equations, differential equations, and optimisation issues, the estimation of fixed points of enriched contractions in Banach spaces has become an important area of study in nonlinear analysis. The class of enriched contractions, which generalise traditional contraction mappings, was developed by Berinde and Păcurar in 2020. This chapter examines the fact that each enriched contraction has a distinct fixed point that may be roughly represented by a suitable Krasnoselskij iterative method. This fundamental result is derived as a corollary of several important results in fixed point theory. In addition, we provide further results concerning the fixed points of asymptotic and local enriched contractions.

1.1 Basic definitions and concepts

Definition 1.1.1. (Metric space) Consider a $\mathcal{X} \neq \varnothing$ and a mapping $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called as metric (or distance function) on $\mathcal{X}$, iff ρ satisfied the following axioms:

(i) $\rho(\varkappa_1, \varkappa_2) \geq 0 \ \forall \ \varkappa_1, \varkappa_2 \in \mathcal{X}$,

(ii) $\rho(\varkappa_1, \varkappa_2) = 0 \iff \varkappa_1 = \varkappa_2$,

(iii) $\rho(\varkappa_1, \varkappa_2) = \rho(\varkappa_2, \varkappa_1) \ \forall \ \varkappa_1, \varkappa_2 \in \mathcal{X}$,

(iv) $\rho(\varkappa_1, \varkappa_2) \leq \rho(\varkappa_1, \varkappa_3) + \rho(\varkappa_3, \varkappa_2) \ \forall \ \varkappa_1, \varkappa_2, \varkappa_3 \in \mathcal{X}$.

The distance between $\varkappa_1$ and $\varkappa_2$ is defined as the value of metric ρ at $(\varkappa_1, \varkappa_2)$, and the pair $(\mathcal{X}, \rho)$ is known as metric space.

Example 1.1.1. The mapping $\rho : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ described by $\rho(\varkappa_1, \varkappa_2) = |\varkappa_1 - \varkappa_2|, \forall \varkappa_1, \varkappa_2 \in \mathbb{R}$ is a metric on $\mathbb{R}$. It is also known as the standard metric on $\mathbb{R}$.

Example 1.1.2. Consider $\mathcal{C}[0, 1]$ as the collection of all continuous functions from $[0, 1]$ to $\mathbb{R}$ with the metric ρ defined by $\rho(\vartheta_1, \vartheta_2) = \sup_{0\leq\upsilon\leq 1} |\vartheta_1(\upsilon) - \vartheta_2(\upsilon)|, \quad \forall \vartheta_1, \vartheta_2 \in \mathcal{C}[0, 1]$.

Example 1.1.3. Let $\mathcal{X} = \mathbb{R}^n$ with Euclidean metric defined as $\rho_e(\varkappa_1, \varkappa_2) = \left(\sum_{i=1}^{n} (\varkappa_1^i - \varkappa_2^i)^2\right)^{\frac{1}{2}}$, where $\varkappa_1, \varkappa_2 \in \mathbb{R}^n$ and $\varkappa_1 = (\varkappa_1^1, \ldots, \varkappa_1^n)$, $\varkappa_2 = (\varkappa_2^1, \ldots, \varkappa_2^n)$.

Definition 1.1.2. (Cauchy sequence) Let $\{\varkappa_n\}$ be a sequence in metric space $(\mathcal{X}, \rho)$. Sequence $\{\varkappa_n\}$ is said to be Cauchy in $(\mathcal{X}, \rho)$, if $\forall\ \varepsilon > 0, \exists\ n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0$, the terms $\varkappa_n, \varkappa_m$ satisfy $\rho(\varkappa_n, \varkappa_m) < \varepsilon$.

Example 1.1.4. If $(\mathcal{X}, \rho)$ is a discrete metric space then every eventually constant sequence in $\mathcal{X}$ is a Cauchy sequence.

Example 1.1.5. Let $\mathcal{X} = (0, 1]$ and ρ is standard metric on the real line. The sequence $\{\varkappa_n\} = \left\{\frac{1}{n}\right\}$ is a Cauchy sequence. However this sequence is not convergent, because $0 \notin (0, 1]$.

Definition 1.1.3. A complete metric space $(\mathcal{X}, \rho)$ is when every Cauchy sequence in $\mathcal{X}$ has a limit exist and belongs to $\mathcal{X}$.

Example 1.1.6. The pair $(\mathbb{R}, \rho)$ is complete if ρ represents the standard metric on $\mathbb{R}$.

Example 1.1.7. The space $(\mathcal{C}[0, 1], \rho)$ defined in Example 1.1.2 is a complete metric space.

Definition 1.1.4. (Fixed point) Conider $\mathcal{X}$ a nonempty set and S a self mapping on $\mathcal{X}$ that is $S : \mathcal{X} \to \mathcal{X}$, then a point x in $\mathcal{X}$ is said to be fixed point in $\mathcal{X}$, if $S(x) = x$.

Example 1.1.8. Let $\mathcal{X} = \mathbb{R}$ and $S : \mathcal{X} \to \mathcal{X}$ be a mapping defined as $S(\varkappa) = \varkappa+1\ \forall, \varkappa \in \mathcal{X}$. Then S has no fixed point in $\mathbb{R}$.

Example 1.1.9. Consider the quadratic polynomial $S(\varkappa) = \varkappa^2 - 3\varkappa+4$. Then clearly $\varkappa = 2$ is a fixed point of S in $\mathbb{R}$

Example 1.1.10. Consider $\mathcal{X} = \mathbb{R}$ and $S : \mathcal{X} \to \mathcal{X}$ a mapping given as

$$S(u) = \begin{cases} 1, & \text{if u=0} \\ u, & \text{otherwise.} \end{cases}$$

$\forall\, u \in \mathbb{R}$. Then S has infinite number of fixed points in $\mathbb{R}$ and also the function is discontinuous at $u = 0$.

Definition 1.1.5. Consider $\mathcal{X}$ a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm on $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ satisfying the following properties for all $\varkappa, \vartheta \in \mathcal{X}$ and all scalars α:

1. $\|\varkappa\| \geq 0$ (non-negativity)
2. $\|\varkappa\| = 0$ iff $\varkappa = 0$ (definiteness)
3. $\|\alpha\varkappa\| = |\alpha| \cdot \|\varkappa\|$ (homogeneity)
4. $\|\varkappa + \vartheta\| \leq \|\varkappa\| + \|\vartheta\|$ (triangle inequality)

A normed space is also a vector space equipped with such a norm.

Example 1.1.11. Consider $\mathcal{X} = \mathbb{R}^n$ the n-dimensional real vector space. Define the norm $\|\cdot\|$ on $\mathcal{X}$ by:

$$\|\varkappa\| = \sqrt{\varkappa_1^2 + \varkappa_2^2 + \cdots + \varkappa_n^2}$$

for any vector $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_n) \in \mathbb{R}^n$. This is called the Euclidean norm (or 2-norm), and it satisfies the norm axioms:

- $\|\varkappa\| \geq 0$ with equality iff $x = 0$
- $\|\alpha\varkappa\| = |\alpha| \cdot \|\varkappa\|$ for any scalar α
- $\|\varkappa + \vartheta\| \leq \|\varkappa\| + \|\vartheta\|$ (Triangle inequality)

Therefore, $(\mathbb{R}^n, \|\cdot\|)$ is a normed linear space. It is also a Banach space.

Definition 1.1.6. A normed linear space is known as Banach space if it is complete .

Example 1.1.12. Let ρ be a metric on $\mathcal{X}$ defined by norm on $\mathcal{X}$, which is given by

$$\rho(\varkappa, \vartheta) = \|\varkappa - \vartheta\| \quad (\varkappa, \vartheta \in \mathcal{X})$$

and is said to be the metric induced by the norm. We can denote a normed space by $(\mathcal{X}, \|\cdot\|)$ or simply by $\mathcal{X}$.

Example 1.1.13. Consider $\mathcal{X} = \ell^q$ the space of all real (or complex) sequences $\varkappa = (\varkappa_1, \varkappa_2, \dots)$ such that

$$\|\varkappa\|_q = \left(\sum_{k=1}^{\infty} |\varkappa_k|^q \right)^{1/q} < \infty.$$

Then $(\ell^q, \|\cdot\|_q)$ is a Banach space.

Definition 1.1.7. An inner product space (or pre-Hilbert space) is a vector space $\mathcal{X}$ over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with an inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ satisfying, for all $\varkappa, \vartheta, z \in \mathcal{X}$ and $a \in \mathbb{K}$:

1. $\langle \varkappa + \vartheta, z \rangle = \langle \varkappa, z \rangle + \langle \vartheta, z \rangle$ (linearity in the first argument)
2. $\langle a\varkappa, \vartheta \rangle = a\langle \varkappa, \vartheta \rangle$ (homogeneity)
3. $\langle \varkappa, \vartheta \rangle = \overline{\langle \vartheta, \varkappa \rangle}$ (conjugate symmetry)
4. $\langle \varkappa, \varkappa \rangle \geq 0$, with equality iff $\varkappa = 0$ (positive-definiteness)

A norm is induced by the inner product on $\mathcal{X}$ defined by

$$\|\varkappa\| = \sqrt{\langle \varkappa, \varkappa \rangle}.$$

Example 1.1.14. Consider $\mathcal{X} = \mathbb{R}^n$, the n-dimensional real vector space. Define the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{X}$ by:

$$\langle \varkappa, \vartheta \rangle = \varkappa_1\vartheta_1 + \varkappa_2\vartheta_2 + \cdots + \varkappa_n\vartheta_n = \sum_{i=1}^{n} \varkappa_i\vartheta_i$$

for any $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_n)$ and $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ in $\mathbb{R}^n$. This inner product satisfies:

- Linearity in the first argument: $\langle a\varkappa + bz, \vartheta \rangle = a\langle \varkappa, \vartheta \rangle + b\langle z, \vartheta \rangle$
- Symmetry: $\langle \varkappa, \vartheta \rangle = \langle \vartheta, \varkappa \rangle$
- Positive definiteness: $\langle \varkappa, \varkappa \rangle \geq 0, \quad$ and $\langle \varkappa, \varkappa \rangle = 0 \Leftrightarrow \varkappa = 0$

Thus, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is an inner product space.

Definition 1.1.8. An inner product space is known as Hilbert space if it is complete .

Remark 1.1.1. A metric on $\mathcal{X}$ given by:

$$\rho(\varkappa, \vartheta) = \|\varkappa - \vartheta\| = \sqrt{\langle \varkappa - \vartheta, \varkappa - \vartheta \rangle}.$$

Hence, inner product spaces are normed spaces, then Hilbert spaces becomes Banach spaces.

Example 1.1.15. Consider $\mathcal{X} = \ell^2$ the space of all real or complex sequences

$$\varkappa = (\varkappa_1, \varkappa_2, \varkappa_3, \dots)$$

such that

$$\|\varkappa\| = \left(\sum_{k=1}^{\infty} |\varkappa_k|^2 \right)^{1/2} < \infty.$$

This is the space of square-summable sequences. Define the inner product on ℓ^2 by:

$$\langle \varkappa, \vartheta \rangle = \sum_{k=1}^{\infty} \varkappa_k \overline{\vartheta_k},$$

where $\varkappa = (\varkappa_k)$ and $\vartheta = (\vartheta_k)$. This inner product satisfies:

- Linearity in the first argument,
- Conjugate symmetry: $\langle \varkappa, \vartheta \rangle = \overline{\langle \vartheta, \varkappa \rangle}$,
- Positive definiteness: $\langle \varkappa, \varkappa \rangle \geq 0, \quad$ and $\langle \varkappa, \varkappa \rangle = 0 \Leftrightarrow \varkappa = 0$.

The norm induced by this inner product is

$$\|\varkappa\| = \left(\sum_{k=1}^{\infty} |\varkappa_k|^2 \right)^{1/2}.$$

Moreover, ℓ^2 is complete with respect to this norm, which means every Cauchy sequence in ℓ^2 converges to a limit in ℓ^2. Therefore, $(\ell^2, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Definition 1.1.9. [4] Let S be a self-mapping on a convex subset C of a linear space. For any $\sigma \in (0, 1)$, define the averaged mapping $S_\sigma : C \to C$ such that

$$S_\sigma \varkappa = (1 - \sigma)\varkappa + \sigma S \varkappa, \quad \forall \varkappa \in C.$$

This mapping satisfies $\text{Fix}(S_\sigma) = \text{Fix}(S)$.

1.2 Contraction mappings in normed linear spaces

Contraction mapping is an essential definition to approximate fixed point in any abstract space. Using sequential criteria and contraction mapping, we reach to our desire result.

Definition 1.2.1. Consider $\mathcal{X}$ a nonempty set and a self-mapping $S : \mathcal{X} \to \mathcal{X}$ then the set of fixed points of S is defined as

$$\text{Fix}(S) = \{\varkappa \in \mathcal{X} : S\varkappa = \varkappa\}.$$

The ith iterate of S, denoted by S^i, is defined recursively as

$$S^0 = I \quad \text{(identity map)}, \quad \text{and} \quad S^i = S \circ S^{i-1}, \quad \text{for } i \geq 1.$$

Definition 1.2.2.

1. Considering $(\mathcal{X}, \|\cdot\|)$ be a linear space that is Normed. Then, contraction occurs when a positive real number $a < 1$ such that for any $\varkappa, \vartheta \in \mathcal{X}$, and the mapping $S : \mathcal{X} \to \mathcal{X}$
$$\|S\varkappa - S\vartheta\| \leq a\|\varkappa - \vartheta\| \quad (a < 1) \tag{1.1}$$

2. Consider $(\mathcal{X}, \|\cdot\|)$ a Normed linear space. Then, a mapping $S : \mathcal{X} \to \mathcal{X}$ is said to be a contractive mapping if there exists a constant $a \in [0, 1)$ such that
$$\|S\varkappa - S\vartheta\| \leq a\|\varkappa - \vartheta\| \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}. \tag{1.2}$$

3. Consider Y a non-empty subset of a real normed linear space . Then, mapping $S : Y \to Y$ is called non-expansive if
$$\|S\varkappa - S\vartheta\| \leq \|\varkappa - \vartheta\| \quad \text{for all } \varkappa, \vartheta \in Y. \tag{1.3}$$

Theorem 1.2.1 (Banach Contraction Theorem [17]). Let $(\mathcal{X}, \rho)$ be a complete metric space

and let $S : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then:

1. S has a unique fixed point $\varkappa \in \mathcal{X}$.

2. For any initial point $\varkappa_0 \in \mathcal{X}$, the Picard iteration

$$\varkappa_{n+1} = S\varkappa_n, \quad n \in \mathbb{N}_0,$$

   converges to $\varkappa$.

3. The sequence satisfies the error estimate

$$\rho(\varkappa_n, \varkappa) \leq \frac{1}{1-\alpha}\alpha^n \rho(\varkappa_0, \varkappa_1), \quad \forall n \in \mathbb{N}_0,$$

   , where $\alpha \in (0, 1)$

Proof. Let $\varkappa_0 \in \mathcal{X}$ be arbitrary, and define the iterative sequence $\{\varkappa_n\}$ by

$$\varkappa_{n+1} = S\varkappa_n, \quad n \in \mathbb{N}_0.$$

We aim to show that $\{\varkappa_n\}$ is a Cauchy sequence and converges to a unique fixed point of S. Using the contraction property of S, we get:

$$\rho(\varkappa_{n+1}, \varkappa_n) = \rho(S\varkappa_n, S\varkappa_{n-1}) \leq \alpha\rho(\varkappa_n, \varkappa_{n-1}) \leq \cdots \leq \alpha^n \rho(\varkappa_1, \varkappa_0).$$

By the triangle inequality and the geometric series sum, for $n > m$, we obtain:

$$\rho(\varkappa_n, \varkappa_m) \leq \sum_{k=m}^{n-1} \rho(\varkappa_{k+1}, \varkappa_k) \leq \sum_{k=m}^{n-1} \alpha^k \rho(\varkappa_1, \varkappa_0) = \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha}\rho(\varkappa_1, \varkappa_0).$$

Since $0 < \alpha < 1$, the right-hand side tends to zero as $m \to \infty$, showing that $\{\varkappa_n\}$ is Cauchy. Since, $\mathcal{X}$ is complete, so $\exists\ \varkappa \in \mathcal{X}$ such that $\varkappa_n \to \varkappa$. Next, we will show that $\varkappa$ is a fixed point of S. Using continuity of S and the contraction condition:

$$\begin{aligned}\rho(\varkappa, S\varkappa) &\leq \rho(\varkappa, \varkappa_m) + \rho(\varkappa_m, S\varkappa) \\ &\leq \rho(\varkappa, \varkappa_m) + \alpha\rho(\varkappa_{m-1}, \varkappa).\end{aligned}$$

As $m \to \infty$, both terms on the right tend to zero. Hence, $\rho(\varkappa, S\varkappa) = 0$, so $S\varkappa = \varkappa$, and $\varkappa$ is a fixed point. To demonstrate the uniqueness, suppose $\varkappa^* \in \mathcal{X}$ be the fixed point other than $\varkappa$ of S. Then:

$$\rho(\varkappa, \varkappa^*) = \rho(S\varkappa, S\varkappa^*) \leq \alpha\rho(\varkappa, \varkappa^*).$$

Since $\alpha < 1$, this implies $\rho(\varkappa, \varkappa^*) = 0$, and thus $\varkappa = \varkappa^*$. Therefore, the fixed point is unique. □

Remark 1.2.1. The converse of Theorem (1.3.1) does not hold true. For example $T\varkappa = 1-\varkappa$ that has a fixed point $\frac{1}{2}$ even though it is not a contraction mapping.

Definition 1.2.3 (Picard Operator). A mapping $S : \mathcal{X} \to \mathcal{X}$ is called a Picard operator only if:

1. $\text{Fix}(S) = \{\mathfrak{p}\}$; i.e., S has a unique fixed point $\mathfrak{p} \in \mathcal{X}$,
2. For every $\varkappa_0 \in \mathcal{X}$, the sequence $S^n\varkappa_0$ converges to $\mathfrak{p}$ as $n \to \infty$.

Example 1.2.1. Consider $C[0,1]$ the Banach space of continuous real-valued functions on $[0,1]$ with the supremum norm $\|\cdot\|_\infty$. Define the operator $T : C[0,1] \to C[0,1]$ by:

$$T(\mathfrak{f}(\varkappa)) = \int_0^1 k(\varkappa, \vartheta)\mathfrak{f}(\vartheta)\, d\vartheta,$$

where:

- $k(\varkappa, \vartheta)$ is continuous on $[0,1] \times [0,1]$,
- $\sup_{\varkappa\in[0,1]} \int_0^1 |k(\varkappa, \vartheta)|\, d\vartheta \leq \alpha < 1$.

Consider $\mathfrak{f}, \mathfrak{h} \in C[0,1]$. Then for all $\varkappa \in [0,1]$:

$$\begin{aligned} |\,T(\mathfrak{f}(\varkappa)) - T(\mathfrak{h}(\varkappa))| &= \left| \int_0^1 k(\varkappa, \vartheta)(\mathfrak{f}(\vartheta) - \mathfrak{h}(\vartheta))\, d\vartheta \right| \\ &\leq \int_0^1 |k(\varkappa, \vartheta)||\mathfrak{f}(\vartheta) - \mathfrak{h}(\vartheta)|\, d\vartheta \\ &\leq \|\mathfrak{f} - \mathfrak{h}\|_\infty \int_0^1 |k(\varkappa, \vartheta)|\, d\vartheta \\ &\leq \alpha\|\mathfrak{f} - \mathfrak{h}\|_\infty. \end{aligned}$$

Thus,

$$\| T\mathfrak{f} - T\mathfrak{h}\|_\infty \leq \alpha \|\mathfrak{f} - \mathfrak{h}\|_\infty.$$

Since $\alpha < 1$, T is a contraction on the Banach space $C[0,1]$. Conclusion: By the Banach Contraction Theorem, T has a unique fixed point $\mathfrak{f}^* \in C[0,1]$ such that

$$\mathfrak{f}^*(\varkappa) = \int_0^1 k(\varkappa, \vartheta)\mathfrak{f}^*(\vartheta)\, d\vartheta,$$

which is the unique solution of the first kind Fredholm-type integral equation.

Remark 1.2.2. The Banach contraction principle states that any mapping $S : \mathcal{X} \to \mathcal{X}$ on a complete metric space $(\mathcal{X}, \rho)$ that satisfies

$$\rho(S\varkappa, S\vartheta) \leq c\,\rho(\varkappa, \vartheta), \quad \forall \varkappa, \vartheta \in \mathcal{X},$$

for some constant $c \in [0,1)$, is a Picard operator.

1.3 Existence of Fixed Points for Enriched Contractions in Banach Spaces

Picard-Banach Contractions and certain non-expansive mappings are incorporated in the broad category of enriched contracts. We demonstrate that any enriched contraction possess a fixed point, and that an appropriate Krasnoselskij iterative strategy may be utilizes to approximate this fixed point.

Definition 1.3.1 ([7]). Consider $(\mathcal{X}, \|\cdot\|)$ a normed linear space. Then, a mapping $S : \mathcal{X} \to \mathcal{X}$ is known as (b,θ)-enriched contraction if there exist constants $b \geq 0$ and $0 \leq \theta < b+1$ such that for all $\varkappa, \vartheta \in \mathcal{X}$,

$$\| b(\varkappa - \vartheta) + S\varkappa - S\vartheta \| \leq \theta \|\varkappa - \vartheta\|.$$

Example 1.3.1. 1. Every contraction S is a $(0,c)$-enriched contraction with contraction constant c , i.e., it satisfies the enriched contraction condition with $b = 0$ and $\theta = c \in [0,1)$.

2. Consider $X = [0, 2]$ equipped with the usual norm, and define $S : \mathcal{X} \to \mathcal{X}$ by $S\varkappa = 2 - \varkappa$ for all $\varkappa \in X$. Although S is an isometry (hence nonexpansive) and not a contraction, it is an enriched contraction. If S were a contraction, so $\exists c \in [0, 1)$ such that

$$|\varkappa - \vartheta| \leq c\,|\varkappa - \vartheta|, \quad \forall \varkappa, \vartheta \in X,$$

which is impossible for $\varkappa \neq \vartheta$. However, S satisfies the enriched contraction condition

$$|(b-1)(\varkappa - \vartheta)| \leq \theta|\varkappa - \vartheta|, \quad \forall \varkappa, \vartheta \in X,$$

for any $b \in (0, 1)$ with $\theta = 1-b \in [0, b+1)$. Thus, S is a $(b, 1-b)$-enriched contraction for all $b \in (0, 1)$. Note that $\mathrm{Fix}(S) = \{1\}$.

Remark 1.3.1. The Picard iteration $\{\varkappa_n\}$ associated with S, that is, the sequence

$$\varkappa_{n+1} = 2 - \varkappa_n, \quad n \geq 0,$$

considering any initial guess $\varkappa_0$ different from 1, the iteration does not converge to the unique fixed point of T.

The aforementioned example demonstrates that standard iterative techniques will not yield a fixed point for enriched contractions. More complex iterative schemes are needed for that, such as the Krasnoselskij iterative approach, which is defined below and offers strong convergence for the class of enriched contractions.

Theorem 1.3.1 ([7]). Consider $(\mathcal{X}, \|\cdot\|)$ a Banach space, and $S : \mathcal{X} \to \mathcal{X}$ a (b, θ)-enriched contraction. Then:

1. The mapping S has a unique fixed point, means $\mathrm{Fix}(S) = \{\varkappa\}$ and $\exists \sigma \in (0, 1]$ such that the sequence $\{\varkappa_n\}_{n=0}^{\infty}$ defined by

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n, \quad n \geq 0,$$

for any $\varkappa_0 \in \mathcal{X}$, the sequence converges to $\varkappa$.

2. The convergence estimate

$$\|\varkappa_{n+i-1} - \varkappa\| \leq c^i \left(\frac{1}{1-c}\right) \|\varkappa_n - \varkappa_{n-1}\|, \quad n, i \in \mathbb{N} \tag{1.4}$$

holds, where $c = \frac{\theta}{b+1}$.

Proof. We consider two separate cases.

Case 1: $b > 0$. Define $\sigma = \frac{1}{b+1}$. Since $0 < \sigma < 1$, the enriched contraction condition (Definition 1.3.1) becomes:

$$\left\| \left(\frac{1}{\sigma} - 1\right)(\varkappa - \vartheta) + S\varkappa - S\vartheta \right\| \leq \theta \|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in \mathcal{X},$$

which is equivalent to

$$\|S_\sigma \varkappa - S_\sigma \vartheta\| \leq c\|\varkappa - \vartheta\|, \quad \text{where } c = \sigma\theta \in (0, 1).$$

Here, S_σ is the averaged mapping defined in Definition 1.1.9. Thus, S_σ is a contraction. The sequence $\{\varkappa_n\}_{n=0}^{\infty}$ generated by

$$\varkappa_{n+1} = (1-\sigma)\varkappa_n + \sigma S\varkappa_n = S_\sigma \varkappa_n,$$

is the Picard iteration for S_σ. Hence,

$$\|\varkappa_{n+1} - \varkappa_n\| \leq c\|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1.$$

From this, we derive the following inequalities:

$$\|\varkappa_{n+m} - \varkappa_n\| \leq c^n \left(\frac{1-c^m}{1-c}\right) \|\varkappa_1 - \varkappa_0\|, \quad n \geq 0,\ m \geq 1,$$

$$\|\varkappa_{n+m} - \varkappa_n\| \leq c \left(\frac{1-c^m}{1-c}\right) \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1,\ m \geq 1.$$

Thus, the cauchy sequence $\{\varkappa_n\}$ converges to some limit $\varkappa \in \mathcal{X}$. From continuity of S_σ, we find $\varkappa = \lim_{n\to\infty} \varkappa_n = \lim_{n\to\infty} S_\sigma \varkappa_n = S_\sigma \varkappa$, so $\varkappa \in \mathrm{Fix}(S_\sigma)$. Suppose there exists another fixed point $\vartheta \neq \varkappa$, then $\|\varkappa - \vartheta\| = \|S_\sigma \varkappa - S_\sigma \vartheta\| \leq c\|\varkappa - \vartheta\| < \|\varkappa - \vartheta\|$, a contradiction. Hence,

$\text{Fix}(S_\sigma) = \{\varkappa\}$. Since $\text{Fix}(S_\sigma) = \text{Fix}(S)$, this proves part (1). Part (2) follows immediately from the convergence of the sequence to $\varkappa$. For part (3), letting $m \to \infty$ in the above inequalities yields

$$\|\varkappa_n - \varkappa\| \leq \frac{c^n}{1-c}\|\varkappa_1 - \varkappa_0\|, \quad \|\varkappa_n - \varkappa\| \leq \frac{c}{1-c}\|\varkappa_n - \varkappa_{n-1}\|.$$

Combining these gives the estimate in (1.4).

Case 2: When $b = 0$. For this case, take $\sigma = 1$, so that $S_\sigma = S$ and $c = \theta$. The iteration becomes the Picard iteration:

$$\varkappa_{n+1} = S\varkappa_n,$$

which converges by similar arguments as in Case 1. □

Remark 1.3.2.

1. The classical Banach contraction principle in Banach spaces emerges as a special case of Theorem 1.3.1 when the parameter $b = 0$.

2. Enriched Kannan contractions, introduced and analyzed in [2], form a distinct class of mappings. While the mapping S discussed in Example 1.3.2(2) satisfies the conditions of both an enriched contraction and an enriched Kannan contraction, it is evident that the class of enriched Kannan contractions is not contained in, nor does it contain, the class of enriched Banach contractions; the two are mutually independent.

1.4 Local and Asymptotic Forms of Enriched Contraction Mappings

The consequences of above theorem is the localised version of fixed point theorem using enriched contraction over the subset of Banach space. We prove that it still holds true for non-self mappings over Banach spaces. Consider an open ball in Banach space $(X, \|\cdot\|)$ and non-self contraction map of B into $\mathcal{X}$ possess important property that it does not displace the centre of ball too far. Then, the following corollary holds true.

Corollary 1.4.1 ([7]). Consider $(\mathcal{X}, \|\cdot\|)$ a Banach space, let $r > 0$, and define the $B := B(\varkappa_0, r) = \{\varkappa \in \mathcal{X} : \|\varkappa - \varkappa_0\| < r\}$. Suppose that $S : B \to \mathcal{X}$ is a (b, θ)-enriched

contraction with parameters $b \geq 0$ and $0 \leq \theta < b+1$. If $\|S\varkappa_0 - \varkappa_0\| < (b+1-\theta)r$, then S has a fixed point in B.

Proof. Select $\epsilon \in (0, r)$ in such a way that $\|S\varkappa_0 - \varkappa_0\| \leq (b+1-\theta)\epsilon$. Since S is a (b, θ)-enriched contraction, we have

$$\|b(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq \theta\|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in B.$$

Case 1: $b > 0$. Define the averaged map

$$S_\sigma := (1-\sigma)I + \sigma S, \quad \text{with } \sigma = \frac{1}{b+1}.$$

As it is proven in Theorem 2.3.1, S_σ is a contraction on B with constant $c = \sigma\theta = \frac{\theta}{b+1} \in [0, 1)$. Moreover,

$$\|S_\sigma\varkappa_0 - \varkappa_0\| = \|(1-\sigma)\varkappa_0 + \sigma S\varkappa_0 - \varkappa_0\| = \sigma\|S\varkappa_0 - \varkappa_0\| \leq \lambda(b+1-\theta)\epsilon = (1-c)\epsilon.$$

Now, let $B_\epsilon := \{\varkappa \in \mathcal{X} : \|\varkappa - \varkappa_0\| \leq \epsilon\} \forall \varkappa \in B_\epsilon$,

$$\begin{aligned}\|S_\sigma\varkappa - \varkappa_0\| &\leq \|S_\sigma\varkappa - S_\sigma\varkappa_0\| + \|S_\sigma\varkappa_0 - \varkappa_0\| \\ &\leq c\|\varkappa - \varkappa_0\| + (1-c)\epsilon \\ &\leq c\epsilon + (1-c)\epsilon = \epsilon.\end{aligned}$$

Thus, $S_\sigma(B_\epsilon) \subseteq B_\epsilon$. Since B_ϵ is a closed, convex, and complete subset of the Banach space $\mathcal{X}$, and S_σ is a contraction, Banach's fixed point theorem suggests that S_σ possess a unique fixed point in B_ϵ, which is also a fixed point of S.

Case 2: $b = 0$. For this case, the enriched contraction condition reduces to the standard contraction:

$$\|S\varkappa - S\vartheta\| \leq \theta\|\varkappa - \vartheta\|, \quad \theta \in [0, 1),$$

and the assumption becomes $\|S\varkappa_0 - \varkappa_0\| < (1-\theta)r$. As before, choose $\epsilon \in (0, r)$ such that $\|S\varkappa_0 - \varkappa_0\| \leq (1-\theta)\epsilon$. Then, using the standard contraction mapping principle, we found that S has a unique fixed point in the closed ball B_ϵ. □

Remark 1.4.1. For a particular case $b = 0$, by 1.4.1, the local version of the contraction

mapping principle is obtained.

There exist certain mappings S which are not contractions, but a certain iteration of it is a contraction.

Example 1.4.1. Let $\mathcal{X} = \mathbb{R}$ and $S : \mathcal{X} \to \mathcal{X}$ be given by

$$S(\varkappa) = \begin{cases} 0, & \text{if } \varkappa \in (-\infty, 1], \\ -\frac{1}{5}, & \text{if } \varkappa \in (1, +\infty). \end{cases} \tag{1.5}$$

In that case, S is not a contraction since it is not continuous, but S^2 is.

Thus, the following theorems gives explanation to above example.

Theorem 1.4.1. [7] Consider $(\mathcal{X}, \rho)$ a complete metric space and $S : \mathcal{X} \to \mathcal{X}$ a mapping. Then, if $\exists$ a positive integer N such that S^N is a contraction, then

$$\text{Fix}(S) = \{\varkappa^*\}.$$

Theorem 1.4.2 ([7]). Consider $(\mathcal{X}, \|\cdot\|)$ a Banach space, and $U : \mathcal{X} \to \mathcal{X}$ be a mapping such that, for some positive integer N, the composition U^N is a (b, θ)-enriched contraction. Then:

1. The set of fixed points of U is a singleton, i.e., $\text{Fix}(U) = \{\varkappa\}$.
2. There exists $\sigma \in (0, 1]$ such that the iteration defined by

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma U^N(\varkappa_n), \quad n \geq 0, \tag{1.6}$$

converges to $\varkappa$ for any initial point $\varkappa_0 \in \mathcal{X}$.

Proof. Let $S := U^N$. We split the proof into two cases.

Case 1: $b > 0$. Set

$$\sigma := \frac{1}{b+1} \in (0, 1).$$

Then the enriched contraction condition implies:

$$\left\| \left(\frac{1}{\sigma} - 1 \right) (\varkappa - \vartheta) + S\varkappa - S\vartheta \right\| \leq \theta \|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in \mathcal{X}.$$

Define the averaged mapping:

$$S_\sigma := (1-\sigma)I + \sigma S.$$

Then it follows that $\|S_\sigma \varkappa - S_\sigma \vartheta\| \leq c\|\varkappa - \vartheta\|$, where $c := \sigma\theta \in (0,1)$. Thus, S_σ is a contraction on $\mathcal{X}$, and the iterative process: $\varkappa_{n+1} = S_\sigma \varkappa_n,\ n \geq 0$, is a standard Picard iteration for a contraction mapping. Hence, the sequence $\{\varkappa_n\}$ converges to a unique fixed point $\varkappa$ of S_σ. To verify uniqueness, assume $\vartheta \neq \varkappa$ is another fixed point of S_σ. Then:

$$\|\varkappa - \vartheta\| = \|S_\sigma \varkappa - S_\sigma \vartheta\| \leq c\|\varkappa - \vartheta\| < \|\varkappa - \vartheta\|,$$

a contradiction. Therefore, $\mathrm{Fix}(S_\sigma) = \{\varkappa\}$. Since $\mathrm{Fix}(S_\sigma) = \mathrm{Fix}(S) = \mathrm{Fix}(U^N)$, we have shown that the fixed point is unique for U^N. Finally, to demonstrate that $\varkappa$ is also a fixed point of U, note that:

$$U^N(U(\varkappa)) = U^{N+1}(\varkappa) = U(U^N(\varkappa)) = U(\varkappa).$$

Thus, $U(\varkappa)$ is also a fixed point of U^N, but since U^N possess a fixed point $\varkappa$ which unique, we conclude that $U(\varkappa) = \varkappa$, so $\varkappa \in \mathrm{Fix}(U)$ and $\mathrm{Fix}(U) = \{\varkappa\}$.

Case 2: $b = 0$. In this case, $S = U^N$ is a standard contraction mapping with constant $\theta \in [0,1)$. The conclusion follows directly from Banach's fixed point theorem using the iteration

$$\varkappa_{n+1} = S\varkappa_n = U^N(\varkappa_n), \quad n \geq 0.$$

This sequence converges to the fixed point $\varkappa$ of U^N which is unique, and the same reasoning as above shows that $\varkappa$ is also a fixed point of U.

□

Chapter 2

Convergence Analysis of Enriched Non-Expansive Mappings in Hilbert Spaces

We examine a new method in this chapter that improves contractive-type mappings using Krasnoselskij averaging, which is a method that is presented here for the first time. Building on this approach, we define and investigate a new class of Hilbert space enriched non-expansive mappings. We establish both strong and weak convergence results and use the Krasnoselskij iterative technique to estimate their fixed points. The breadth and depth of this new class of contractive mappings are demonstrated by a number of instances.

2.1 Prerequisites

Definition 2.1.1. [19] A normed linear space $\mathcal{X}$ is called as uniformly convex if and only if, given $\epsilon > 0, \exists \delta(\epsilon) > 0$ such that

$$\left\| \frac{\varkappa + \vartheta}{2} \right\| \leq 1 - \delta(\epsilon) \text{ whenever } \|\varkappa - \vartheta\| \geq \epsilon \text{ and } \|\varkappa\| = \|\vartheta\| = 1.$$

Example 2.1.1. Consider $L^q(\mathcal{X})$ the space of measurable functions $f : \mathcal{X} \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$\|f\|_q = \left(\int_{\mathcal{X}} |f(t)|^q \, dt \right)^{1/q} < \infty.$$

Then, the space $(L^q(\mathcal{X}), \|\cdot\|_q)$ is uniformly convex for all q with $1 < q < \infty$.

Definition 2.1.2. [19] A normed linear space is locally uniformly convex (l.u.c.) iff, given

$\epsilon > 0$ and an element $\mathcal{X}$ with $\|\varkappa\| = 1$, there exists $\delta(\epsilon, \varkappa) > 0$ such that

$$\left\| \frac{\varkappa + \vartheta}{2} \right\| \leq 1 - \delta(\epsilon, x) \text{ whenever } \|\varkappa - \vartheta\| \geq \epsilon \text{ and } \|\vartheta\| = 1.$$

Example 2.1.2. The space $C[0,1]$ with the sup norm is not locally uniformly convex, but the space $C[0,1]$ with the norm

$$\|f\| = \left(\int_0^1 |f(t)|^q \, dt \right)^{1/q} + |f(0)|$$

An example of a locally uniformly convex normed space that is not uniformly convex globally is $1 < q < \infty$.

Definition 2.1.3 (Strict convexity of normed spaces [19]). A normed linear space $\mathcal{X}$ is called strictly convex if the following condition holds:

$$\|\varkappa + \vartheta\| = \|\varkappa\| + \|\vartheta\| \quad \text{implies} \quad \varkappa = t\vartheta \text{ for some } t > 0, \tag{2.1}$$

whenever $\varkappa, \vartheta \in \mathcal{X}$ are both nonzero. In other words, the norm is strictly convex if equality in the triangle inequality implies that the two vectors are positively linearly dependent. It follows from the standard definitions that:

- Uniform convexity $\Rightarrow$ local uniform convexity;
- Local uniform convexity $\Rightarrow$ strict convexity.

Example 2.1.3. The space ℓ^q for $1 < q < \infty$. The sequence space

$$\ell^q = \left\{ \varkappa = (\varkappa_1, \varkappa_2, \dots) : \|\varkappa\|_q = \left(\sum_{k=1}^{\infty} |\varkappa_k|^q \right)^{1/q} < \infty \right\}$$

is strictly convex for $1 < q < \infty$.

Example 2.1.4. The Sobolev space $W^{1,q}(\mathcal{X})$ for $1 < q < \infty$. This space of functions with integrable derivatives is defined by

$$W^{1,q}(\mathcal{X}) = \{u \in L^q(\mathcal{X}) : Du \in L^q(\mathcal{X})\}$$

with norm

$$\|u\|_{W^{1,q}} = \left(\|u\|_q^q + \|Du\|_q^q\right)^{1/q},$$

and is also strictly convex when $1 < q < \infty$.

Definition 2.1.4 (Compact Linear Operator). Consider $\mathcal{X}$ and $\mathcal{Y}$ normed vector spaces. A linear operator $S : \mathcal{X} \to \mathcal{Y}$ is said to be a compact linear operator if $\forall$ bounded subset $M \subset \mathcal{X}$, the image $S(M)$ is relatively compact in $\mathcal{Y}$, which means, the closure $\overline{S(M)}$ is compact.

Example 2.1.5 (Integral Operator as a Compact Operator). Consider the integral equation:

$$(S - \sigma I)\varkappa(s) = \vartheta(s), \text{ where } (S\varkappa)(s) := \int k(s,t)\varkappa(t)\,dt. \tag{2.2}$$

Here, $\sigma \in \mathbb{C}$ is a parameter, and ϑ and $k(s,t)$ are given functions satisfying suitable regularity and integrability conditions. If the kernel k is continuous on a compact domain, then S defines a compact linear operator on an appropriate Banach space such as $C[a,b]$ or $L^2[a,b]$.

Remark 2.1.1. Non-expansive mappings can be regarded as a limiting case of Picard-Banach contractions. In the context of normed linear spaces, a Picard-Banach contraction is a mapping $S : K \to K$ satisfying the condition:

$$\|S\varkappa - S\vartheta\| \leq c\|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in K,$$

where $c \in [0,1)$ is the contraction coefficient.

Theorem 2.1.1. [18] Consider C a closed subset of $\mathcal{X}$, and $\mathcal{X}$ a uniformly convex Banach space. The mapping defined by $S(C)$ is compact if $S : C \to C$ is non-expansive.

$$S_{\frac{1}{2}}\varkappa = \frac{1}{2}\varkappa + \frac{1}{2}S\varkappa$$

possesses the characteristic that its iteration sequence always converges to S. Given the same fixed points of S and $S_{\frac{1}{2}}$, the limit of a convergent sequence provided by

$$\varkappa_{n+1} = \frac{1}{2}\varkappa_n + \frac{1}{2}S\varkappa_n, \quad n \geq 0$$

is necessarily a fixed point of S.

2.2 Enriched Non-expansive Mappings in Hilbert Spaces

Enriched non-expansive mappings are the generalization of non-expansive mappings in Hilbert spaces. It also includes enriched contractions which makes it more useful to derive results even in abstract spaces where convergence of results is not trivial.

Definition 2.2.1. [6] Consider $(\mathcal{X}, \|\cdot\|)$ a linear normed space. Then, a mapping $S : \mathcal{X} \to \mathcal{X}$ is called an enriched nonexpansive mapping if $\exists b \geq 0$ such that

$$\|b(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq (b+1)\|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in X. \tag{2.3}$$

We will furthermore refer to S as a b-enriched nonexpansive mapping in order to denote the constant involved in 2.3.

Remark 2.2.1. Each non-expansive mapping S is clearly a 0-enriched mapping, meaning that it satisfies 2.3 with $b = 0$,.

$$\|S\varkappa - S\vartheta\| \leq \|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in \mathcal{X}. \tag{2.4}$$

Example 2.2.1. Let the mapping T be defined as:

$$T(\varkappa) = \begin{cases} 1 - x, & \text{if } \varkappa \in [0, 1), \\ 0.4, & \text{if } \varkappa = 1. \end{cases}$$

This function is discontinuous at $\varkappa = 1$. Consequently, it does not meet the criteria for a classical non-expansive mapping. For instance, if we consider a sequence $\varkappa_n \to 1$, we observe that $T(\varkappa_n) = 1 - \varkappa_n \to 0$, whereas $T(1) = 0.4$. This discrepancy violates the continuity typically required for classical non-expansiveness.

However, T might fulfill an enriched non-expansive condition, where the average distance terms compensate for the discontinuity at $\varkappa = 1$. Let's illustrate this heuristically by examining $\varkappa = 1$ and $\vartheta = 0.9$: We have $T(1) = 0.4$ and $T(0.9) = 0.1$. The distance between the images is $|T(1) - T(0.9)| = |0.4 - 0.1| = 0.3$. Now, consider an enriched parameter

$k = 0.5$. The average of the points is $0.5\varkappa + 0.5\vartheta = 0.5(1) + 0.5(0.9) = 0.5 + 0.45 = 0.95$. The terms for the right-hand side (RHS) of an enriched condition would involve distances like $|(0.5\varkappa + 0.5\vartheta) - T(\vartheta)|$ and $|(0.5\vartheta + 0.5\varkappa) - T(\varkappa)|$. Specifically: $|0.95 - T(0.9)| = |0.95 - 0.1| = 0.85$. $|0.95 - T(1)| = |0.95 - 0.4| = 0.55$. If the enriched condition is of the form involving b times the sum of these average distances, the RHS would be $b(0.85 + 0.55) = b(1.4)$. For the enriched condition to hold in this specific instance, we would need $0.3 \leq b(1.4)$, implying $b \geq 0.3/1.4 \approx 0.214$. Since we can choose a b value (e.g., $b = 0.22$) such that $0.214 \leq b < 1$ (as required by some enriched conditions), the enriched condition can be satisfied even though the map is not continuous and thus not classically non-expansive.

Definition 2.2.2. [20] A mapping $S : C \to H$ is defined as demicompact if, for any bounded sequence $\{u_n\}$ in a subset C of a Hilbert space H such that $\{Su_n - u_n\}$ converges strongly, $\exists$ a strongly convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Example 2.2.2. Let H be a Hilbert space (e.g., l^2), and define the operator $S : H \to H$ by

$$S\varkappa = \varkappa + K\varkappa,$$

where:

- K is a compact linear operator (e.g., an integral operator 2.1.5),
- I is not compact, but of course bounded.

So $S = I + K$ is not compact, because the identity operator is not compact. So S is demicompact operator.

Definition 2.2.3. [10] Let H be a Hilbert space and $\mathcal{C}$ a closed convex subset of H. A mapping $S : \mathcal{C} \to \mathcal{C}$ is considered asymptotically regular (on $\mathcal{C}$) if, for every $\varkappa \in \mathcal{C}$, the following holds:

$$\|S^{n+1}\varkappa - S^n\varkappa\| \to 0 \quad \text{as } n \to \infty.$$

Example 2.2.3. Let $H = \ell^2$, the Hilbert space of square-summable sequences, and let $C = \ell^2$, a closed convex subset. Define the operator $S : \ell^2 \to \ell^2$ by

$$S^n\varkappa = \left(1 - \frac{1}{n+1}\right)\varkappa_n.$$

This operator is:

- Nonexpansive, since for all $\varkappa, \vartheta \in \ell^2$, $\|S\varkappa - S\vartheta\| \leq \|\varkappa - \vartheta\|$,
- Asymptotically regular, because $\|S^{n+1}\varkappa - S^n\varkappa\| \to 0 \quad \text{as } n \to \infty$.

Thus, S is an example of an asymptotically regular mapping on a closed convex subset of a Hilbert space.

Lemma 2.2.1. [10] Consider H a Hilbert space and C a closed convex subset of H. If the mapping $U : C \to C$ is nonexpansive and $\text{Fix}(U) \neq \emptyset$, then for any given $\sigma \in (0, 1)$, the mapping $U_\sigma = \sigma I + (1 - \sigma)U$ maps C into C, possess the same fixed points as U, and is asymptotically regular.

Now, the above definitions and lemma will be use to prove our following main result.

Theorem 2.2.1. Consider $S : C \to C$ a b-enriched nonexpansive and demicompact mapping, where C is a bounded, closed, and convex subset of a Hilbert space H. Then for any initial point $\varkappa_0 \in C$, the Krasnoselskij iteration $\{\varkappa_n\}_{n=0}^{\infty}$ defined by

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n, \quad \text{for some } \sigma \in (0, 1),$$

converges strongly to a fixed point of S. Moreover, the fixed point set $\text{Fix}(S)$ is nonempty and convex.

Proof. Since S is b-enriched nonexpansive, $\exists$ a constant $b \in [0, 1)$ such that

$$\|b(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq (1 + b)\|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in C.$$

Define $\sigma = \frac{1}{1+b}$ so that $b = \frac{1}{\sigma} - 1$, which implies $\sigma \in (0, 1]$. The above inequality becomes:

$$\|(1 - \sigma)(\varkappa - \vartheta) + \sigma(S\varkappa - S\vartheta)\| \leq \|\varkappa - \vartheta\|, \quad \forall \varkappa, \vartheta \in C.$$

Define the averaged operator $S_\sigma : C \to C$ by $S_\sigma\varkappa = (1-\sigma)\varkappa+\sigma S\varkappa$. Then S_σ is nonexpansive $\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq \|\varkappa - \vartheta\|, \ \forall \varkappa, \vartheta \in C$. By the Browder–Göhde–Kirk fixed point theorem (cf. Theorem 3.1 in [5]), since C is closed, convex, bounded in a Hilbert space and S_σ is nonexpansive, we conclude that $\text{Fix}(S_\sigma) \neq \emptyset$. Since $\text{Fix}(S) = \text{Fix}(S_\sigma)$ (from Lemma 2.2.1),

it follows that $\mathrm{Fix}(S)$ is also nonempty. To show that $\mathrm{Fix}(S_\sigma)$ is convex, let $\varkappa, \vartheta \in \mathrm{Fix}(S_\sigma)$ and let $\alpha \in [0,1]$. Define $u_\alpha = (1-\alpha)\varkappa + \alpha\vartheta$. We want to show that $u_\alpha \in \mathrm{Fix}(S_\sigma)$. Since S_σ is nonexpansive

$$\|S_\sigma u_\alpha - \varkappa\| = \|S_\sigma u_\alpha - S_\sigma \varkappa\| \leq \|u_\alpha - \varkappa\|, \tag{2.5}$$

$$\|S_\sigma u_\alpha - \vartheta\| \leq \|u_\alpha - \vartheta\|. \tag{2.6}$$

Adding (2.5) and (2.6) gives:

$$\begin{aligned}\|\varkappa - \vartheta\| &\leq \|\varkappa - S_\sigma u_\alpha\| + \|S_\sigma u_\alpha - \vartheta\| \\ &\leq \|u_\alpha - \varkappa\| + \|u_\alpha - \vartheta\| \\ &= \alpha\|\varkappa - \vartheta\| + (1-\alpha)\|\varkappa - \vartheta\| = \|\varkappa - \vartheta\|.\end{aligned}$$

Hence, equality must hold throughout, and this implies

$$\varkappa - S_\sigma u_\alpha = a(\varkappa - u_\alpha), \quad \vartheta - S_\sigma u_\alpha = b(\vartheta - u_\alpha),$$

for some $a, b \in [0,1]$. Now set $\alpha = 1$, yielding $a = 1$, and $\alpha = 0$, yielding $b = 1$. Therefore, $S_\sigma u_\alpha = u_\alpha$, which implies $u_\alpha \in \mathrm{Fix}(S_\sigma)$, and so $\mathrm{Fix}(S_\sigma)$ and hence $\mathrm{Fix}(S)$ is convex. Now consider the Krasnoselskij iteration:

$$\varkappa_{n+1} = (1-\alpha)\varkappa_n + \alpha S_\sigma \varkappa_n.$$

This sequence stays in C and is bounded. Define the operator $U_\alpha = (1-\alpha)I + \alpha S_\sigma$. Then U_α is nonexpansive, and by Lemma 2.2.1, asymptotically regular:

$$\|\varkappa_n - U_\alpha \varkappa_n\| \to 0 \quad \text{as } n \to \infty.$$

Note that

$$U_\alpha \varkappa - \varkappa = \alpha(S_\sigma \varkappa - \varkappa) = \alpha\sigma(S\varkappa - \varkappa),$$

so $\|\varkappa_n - S_\sigma \varkappa_n\| \to 0$. Since S is demicompact, so is S_σ. Therefore, there exists a subsequence $\{\varkappa_{n_k}\}$ converging strongly to some $\mathfrak{q} \in C$. Continuity of S_σ implies $S_\sigma \varkappa_{n_k} \to S_\sigma \mathfrak{q}$, and hence

$\varkappa_{n_k} - S_\sigma \varkappa_{n_k} \to \mathfrak{q} - S_\sigma \mathfrak{q}$, but the left-hand side also tends to 0, so $\mathfrak{q} = S_\sigma \mathfrak{q}$, i.e., $\mathfrak{q} \in \mathrm{Fix}(S)$. Finally, the inequality $\|\varkappa_{n+1} - \mathfrak{q}\| \leq \|\varkappa_n - \mathfrak{q}\|$ shows that the full sequence $\{\varkappa_n\}$ converges strongly to $\mathfrak{q} \in \mathrm{Fix}(S)$. □

2.3 On Weak Convergence of Enriched Nonexpansive Mappings in Hilbert Spaces

The asymptotic regularity of the averaged operator derived from a nonexpansive mapping S plays a key role in establishing the main result in the previous section. When combined with the demicompactness of S, it ensures strong convergence. However, in the absence of demicompactness, asymptotic regularity alone is generally insufficient to guarantee strong convergence of the Krasnoselskij iteration sequence $\{\varkappa_n\}_{n=0}^{\infty}$, though it still implies weak or limited convergence.

Theorem 2.3.1. Let H be a Hilbert space and $\mathcal{C}$ a bounded, closed, and convex subset of H. Assume $S : \mathcal{C} \to \mathcal{C}$ is an enriched non-expansive operator with a unique fixed point $\mathfrak{p}$. Given an arbitrary initial point $\varkappa_0 \in \mathcal{C}$ and a constant $\sigma \in (0, 1)$, the Krasnoselskij iterative sequence is given by:

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S(\varkappa_n), \quad n \geq 0. \tag{2.7}$$

Under these conditions, the sequence $\{\varkappa_n\}$ converges in the weak topology to $\mathfrak{p}$.

Proof. Define the averaged operator

$$S_\sigma \varkappa := (1 - \sigma)\varkappa + \sigma S\varkappa,$$

for some $\sigma \in (0, 1)$. By properties of enriched nonexpansive mappings, S_σ is nonexpansive and $\mathrm{Fix}(S_\sigma) = \mathrm{Fix}(S) = \{\mathfrak{p}\}$. Consider the subsequence $\{\varkappa_{n_j}\}$ generated by

$$\varkappa_{n_j+1} = (1 - \alpha)\varkappa_{n_j} + \alpha S_\sigma \varkappa_{n_j}, \quad j \geq 0.$$

Suppose $\varkappa_{n_j} \rightharpoonup \mathfrak{p}_0$ weakly. Since $\mathfrak{p}_0$ is a fixed point of

$$U_\alpha := (1 - \alpha)I + \alpha S_\sigma,$$

it follows that $\mathfrak{p}_0 \in \mathrm{Fix}(U_\alpha) = \mathrm{Fix}(S_\sigma) = \mathrm{Fix}(S)$, and by uniqueness, $\mathfrak{p}_0 = \mathfrak{p}$. Assume, for contradiction, that the sequence $\{\varkappa_{n_j}\}$ does not converge weakly to $\mathfrak{p}$. Using asymptotic regularity and nonexpansiveness of U_α, we have

$$\|\varkappa_{n_j} - S_\sigma \varkappa_{n_j}\| \to 0 \quad \text{as } j \to \infty.$$

Next,

$$\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\| \leq \|U_\alpha \varkappa_{n_j} - U_\alpha \mathfrak{p}_0\| + \|\varkappa_{n_j} - U_\alpha \varkappa_{n_j}\| \leq \|\varkappa_{n_j} - \mathfrak{p}_0\| + \|\varkappa_{n_j} - U_\alpha \varkappa_{n_j}\|,$$

which implies

$$\limsup_{j\to\infty} \big(\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\| - \|\varkappa_{n_j} - \mathfrak{p}_0\|\big) \leq 0.$$

By expanding the square norm,

$$\begin{aligned}\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\|^2 &= \|(\varkappa_{n_j} - \mathfrak{p}_0) + (\mathfrak{p}_0 - U_\alpha \mathfrak{p}_0)\|^2 \\ &= \|\varkappa_{n_j} - \mathfrak{p}_0\|^2 + \|\mathfrak{p}_0 - U_\alpha \mathfrak{p}_0\|^2 + 2\langle \varkappa_{n_j} - \mathfrak{p}_0, \mathfrak{p}_0 - U_\alpha \mathfrak{p}_0\rangle.\end{aligned}$$

Since $\varkappa_{n_j} \rightharpoonup \mathfrak{p}_0$, the last inner product tends to zero, so

$$\lim_{j\to\infty} \big(\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\|^2 - \|\varkappa_{n_j} - \mathfrak{p}_0\|^2\big) = \|\mathfrak{p}_0 - U_\alpha \mathfrak{p}_0\|^2.$$

On the other hand,

$$\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\|^2 - \|\varkappa_{n_j} - \mathfrak{p}_0\|^2 = \big(\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\| - \|\varkappa_{n_j} - \mathfrak{p}_0\|\big)\big(\|\varkappa_{n_j} - U_\alpha \mathfrak{p}_0\| + \|\varkappa_{n_j} - \mathfrak{p}_0\|\big).$$

Given that $\mathcal{C}$ is bounded, the second factor is bounded. Combining these limits yields

$$\|\mathfrak{p}_0 - U_\alpha \mathfrak{p}_0\| = 0, \text{ or equivalently, } U_\alpha \mathfrak{p}_0 = \mathfrak{p}_0.$$

Therefore, $\mathfrak{p}_0 \in \mathrm{Fix}(U_\alpha) = \mathrm{Fix}(S_\sigma) = \mathrm{Fix}(S) = \{\mathfrak{p}\}$, showing that every weak limit point of $\{\varkappa_n\}$ is $\mathfrak{p}$. Hence, the entire sequence converges weakly to $\mathfrak{p}$.

□

Chapter 3

Approximation Methods for Fixed Points of Enriched Chatterjea Contractions in Banach Spaces

In this Chapter, we study a new category of mappings termed enriched Chatterjea contractions which was introduced by Berinde, for which we establish fixed point results and a convergence theorem concerning the Krasnoselskij iterative method in the context of Banach spaces. This framework is then broadened to encompass a more general class, referred to as enriched Chatterjea-type mappings. Enriched Chatterjea contractions allow for convergence guarantees under weaker conditions than classical contractions. In enriched contraction the definition uses distance between pre-images of function to bound distance between images to locate a fixed point. But, it performs way too many number of iterations to locate a fixed point numerically. It also includes distance between pre-images and images of function to bound distance between images to reduce the number of iterations locate a fixed point more precisely. To highlight the expressive power of this new family and to explore its connections with existing contractive mapping classes, we provide illustrative examples. Additionally, numerical simulations are carried out to demonstrate the practical utility and effectiveness of the proposed theoretical developments.

3.1 Mathematical Background

Consider $(\mathcal{X}, \rho)$ a metric space and $S : \mathcal{X} \to \mathcal{X}$ a mapping. Consider two distinct points $\varkappa, \vartheta \in \mathcal{X}$. For which $\exists$ six displacements:

$$\rho(S\varkappa, S\vartheta), \quad \rho(\varkappa, \vartheta), \quad \rho(\varkappa, S\varkappa), \quad \rho(\vartheta, S\vartheta), \quad \rho(\varkappa, S\vartheta), \quad \text{and} \quad \rho(\vartheta, S\varkappa). \tag{3.1}$$

To attain a fixed point result for such a mapping S, it is typical to use two or more displacements from the list (3.1). In 1922, the first metric fixed point theorem was introduced by Banach, originally in the setting of a Banach space, and later extended to complete metric spaces by Caccioppoli [11]. It involves the first two displacements from (3.1):

$$\rho(S\varkappa, S\vartheta) \leq c \cdot \rho(\varkappa, \vartheta), \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{3.2}$$

where c is a constant with $0 \leq c < 1$. It follows directly from (3.2) that any Banach contraction is continuous. The contraction mapping principle is therefore one of the most fundamental tools in nonlinear analysis, due to the simplicity and broad applicability of the contraction condition (3.2). In 1986, Kannan [15, 16] developed a fixed point theorem based on a contractive condition involving three displacements from the list (3.1), which yields the same conclusion as the Picard–Banach contraction principle:

$$\rho(S\varkappa, S\vartheta) \leq a\left[\rho(\varkappa, S\varkappa) + \rho(\vartheta, S\vartheta)\right], \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{3.3}$$

where a is a constant, $a \in \left[0, \frac{1}{2}\right)$. Note that a Kannan contraction is generally not continuous, as illustrated by the following examples.

Example 3.1.1. [15] The map $S : [0, 1] \to [0, 1]$ is defined by

$$S(\varkappa) = \begin{cases} \frac{\varkappa}{4} & \text{if } 0 \leq \varkappa < \frac{1}{2}, \\ \frac{\varkappa}{5} & \text{if } \frac{1}{2} \leq \varkappa \leq 1. \end{cases}$$

It is a discontinuous map with 0 as the unique fixed point. For $\varkappa, \vartheta \in [0, 1]$, one can easily

finds that

$$|S(\varkappa) - S(\vartheta)| \leq \frac{4}{9}\left(|\varkappa - S(\varkappa)| + |\vartheta - S(\vartheta)|\right).$$

Example 3.1.2. The map $S : [0,1] \to [0,1]$ is defined by

$$S\varkappa = \frac{\varkappa}{3}, \quad \forall \varkappa \in [0,1]$$

The above mapping satisfies the Banach contraction condition but fails for $\varkappa = \frac{1}{3}$ and $\vartheta = 0$ under the Kannan mapping condition.

However, for $c < \frac{1}{3}$, a Banach contraction is also a Kannan contraction. Making use of triangle inequality and 3.2 , we obtain

$$\rho(S\varkappa, S\vartheta) \leq c\left[\rho(\varkappa, S\varkappa) + \rho(S\varkappa, S\vartheta) + \rho(S\vartheta, \vartheta)\right]$$

$$\Leftrightarrow \quad \rho(S\varkappa, S\vartheta) \leq \frac{c}{1-c}\left[\rho(\varkappa, S\varkappa) + \rho(\vartheta, S\vartheta)\right],$$

and since $\frac{c}{1-c} < \frac{1}{2}$ for $c < \frac{1}{3}$, inequality 3.3 holds for all $\varkappa, \vartheta \in \mathcal{X}$. The Kannan contraction [14] characterizes the completeness of the space, whereas the Banach contraction does not. In 1972, [12] formulated the following dual counterpart to the Kannan contraction condition:

$$\rho(S\varkappa, S\vartheta) \leq b\left[\rho(\varkappa, S\vartheta) + \rho(\vartheta, S\varkappa)\right], \quad \forall \varkappa, \vartheta \in X, \tag{3.4}$$

where b is a constant with $b \in \left[0, \frac{1}{2}\right)$. Then S is a Picard operator.

3.2 Estimating Enriched Chatterjea Mappings' Fixed Points

Enriched Chatterjea contraction is generalization of Chatterjea contraction, which makes it more useful due to its wide range of applications in mathematics as compare to it's classical Chatterjea contraction.

Definition 3.2.1. [8] Let $(\mathcal{X}, \|\cdot\|)$ be a linear normed space. A mapping $S : \mathcal{X} \to \mathcal{X}$ is said to be an enriched Chatterjea mapping if there exist $b \in [0, 1/2)$ and $k \in [0, +\infty)$ such that,

for all $\varkappa, \vartheta \in X$

$$\|k(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq b[\|(k+1)(\varkappa - \vartheta) + \vartheta - S\vartheta\| + \|(k+1)(\vartheta - \varkappa) + \varkappa - S\varkappa\|]. \quad (3.5)$$

We call S as (k, b)-enriched Chatterjea mapping

Example 3.2.1. [8] Consider the space $\mathcal{X} = [0, 1]$ equipped with its standard norm, and define the mapping $S : \mathcal{X} \to \mathcal{X}$ by

$$S\varkappa = 1 - \varkappa, \quad \forall \varkappa \in [0, 1].$$

The operator S is non-expansive, but it has been shown that it is neither a Banach contraction nor a Kannan mapping. Moreover, S functions as an enriched Chatterjea mapping, not a basic Chatterjea mapping. For S to be a Chatterjea mapping, there would need to exist some $b \in [0, 1/2)$ such that for all $\varkappa, \vartheta \in [0, 1]$:,

$$|\varkappa - \vartheta| \leq 2b \cdot |\varkappa + \vartheta - 1|.$$

This leads to the contradiction $1 \leq 0$ for $\varkappa = 0$ and $\vartheta = 1$. In this case, the enriched Chatterjea condition (3.5) is equivalent to

$$|(k-1)(\varkappa - \vartheta)| \leq b\left[|(k+1)\varkappa - (k-1)\vartheta - 1| + |(k+1)\vartheta - (k-1)\varkappa - 1|\right]. \quad (3.6)$$

Note that

$$\begin{aligned} 2k|\varkappa - \vartheta| &= \left|\big((k+1)\varkappa - (k-1)\vartheta - 1\big) - \big((k+1)\vartheta - (k-1)\varkappa - 1\big)\right| \\ &\leq |(k+1)\varkappa - (k-1)\vartheta - 1| + |(k+1)\vartheta - (k-1)\varkappa - 1|. \end{aligned}$$

For (3.7) to hold for all $\varkappa, \vartheta \in [0, 1]$, it is necessary that $\frac{|k-1|}{2k} \leq b$, for some $b \in [0, 1/2)$. One possibility is to have

$$\frac{1}{2} < k \leq 1 \quad \text{and} \quad \frac{1-k}{2k} \leq b < \frac{1}{2}.$$

By setting $\frac{1-k}{2k} = b$, one obtains

$$k = \frac{1}{2b+1},$$

and therefore, for any $b \in [0, 1/2)$, S is a $\left(\frac{1}{2b+1}, b\right)$-enriched Chatterjea mapping with $\text{Fix}(S) = \left\{\frac{1}{2}\right\}$.

Remark 3.2.1.

1. We can notice in Example 3.2.1 that Picard iteration $\varkappa_n$ associated to S, (i.e., $\varkappa_{n+1} = 1 - \varkappa_n, n \geq 0$) fails to converge to 0, which is the unique fixed point of S, regardless of the initial point $\varkappa_0$.

2. Same situation can be seen in case of non-expansive mappings. So, we require more elaborative iterative methods to approximate the fixed points associated with enriched Chatterjea mappings.

3. This problem can be solve using Krasnoselski's iterative method.

Now, move on to our main result.

Theorem 3.2.1. [8] Given a Banach space $(\mathcal{X}, \|\cdot\|)$ and a mapping $S : \mathcal{X} \to \mathcal{X}$ that is a (k, b)-enriched Chatterjea mapping, it follows that:

1. There exists a unique fixed point $\varkappa$ for S, so $\text{Fix}(S) = \{\varkappa\}$. Moreover, for any $\varkappa_0 \in \mathcal{X}$, a $\sigma \in (0, 1]$ can be found such that the iterative process $\{\varkappa_n\}_{n=0}^{\infty}$, given by

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n, \quad n \geq 0, \tag{3.7}$$

converges to $\varkappa$.

2. The subsequent estimate is valid:

$$\|\varkappa_{n+i-1} - \varkappa\| \leq \frac{\delta^i}{1-\delta}\|\varkappa_n - \varkappa_{n-1}\|, \quad n, i \in \mathbb{N} \tag{3.8}$$

where $\delta = \frac{b}{1-b}$.

Proof. If $k > 0$ in (3.5), then, let

$$\sigma = \frac{1}{k+1}. \tag{3.9}$$

Since $0 < \sigma < 1$, it follows that (3.5) becomes

$$\left\| \left(\frac{1}{\sigma} - 1 \right) (\varkappa - \vartheta) + S\varkappa - S\vartheta \right\| \leq b \left[\left\| \frac{1}{\sigma}(\varkappa - \vartheta) + \vartheta - S\vartheta \right\| + \left\| \frac{1}{\sigma}(\vartheta - \varkappa) + \varkappa - S\varkappa \right\| \right], \quad \forall \varkappa, \vartheta \in \mathcal{X} \tag{3.10}$$

which is equivalent to

$$\|(1-\sigma)(\varkappa-\vartheta)+\sigma(S\varkappa-S\vartheta)\| \leq b \left[\|\varkappa - \vartheta + \sigma(\vartheta - S\vartheta)\| + \|\vartheta - \varkappa + \sigma(\varkappa - S\varkappa)\| \right], \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{3.11}$$

and can be written more simply as

$$\|T_\sigma \varkappa - T_\sigma \vartheta\| \leq b \left[\|\varkappa - T_\sigma \vartheta\| + \|\vartheta - T_\sigma \varkappa\| \right], \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{3.12}$$

with $b \in [0, 1/2)$. From (1.1.9), the iterative sequence $\{\varkappa_n\}_{n=0}^{\infty}$, which is the Picard iteration for T_σ, is defined by (3.7), or can be equivalently written as:

$$\varkappa_{n+1} = T_\sigma \varkappa_n, \quad n \geq 0. \tag{3.13}$$

Taking $\varkappa = \varkappa_n$ and $\vartheta = \varkappa_{n-1}$ in (3.12), we get

$$\|\varkappa_{n+1} - \varkappa_n\| \leq b \left(\|\varkappa_n - \varkappa_{n-1}\| + \|\varkappa_{n-1} - \varkappa_{n+1}\| \right) \leq b \left(\|\varkappa_{n-1} - \varkappa_n\| + \|\varkappa_n - \varkappa_{n+1}\| \right), \tag{3.14}$$

which implies

$$\|\varkappa_{n+1} - \varkappa_n\| \leq \frac{b}{1-b} \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{3.15}$$

Setting $\delta = \frac{b}{1-b}$, since $0 < b < \frac{1}{2}$, we have $0 < \delta < 1$. Hence, the sequence $\{\varkappa_n\}$ satisfies

$$\|\varkappa_{n+1} - \varkappa_n\| \leq \delta \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{3.16}$$

From (3.16), the following two estimates hold:

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \delta^n \left(\frac{1 - \delta^m}{1 - \delta} \right) \|\varkappa_1 - \varkappa_0\|, \quad n \geq 0, m \geq 1, \tag{3.17}$$

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \delta \left(\frac{1 - \delta^m}{1 - \delta} \right) \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1, m \geq 1. \tag{3.18}$$

By (3.17), $\{\varkappa_n\}$ converges in the Banach space $(\mathcal{X}, \|\cdot\|)$ because it is a cauchy sequence. Let

$$\varkappa = \lim_{n\to\infty} \varkappa_n \tag{3.19}$$

We show that $\varkappa$ is a fixed point of T_σ:

$$\|\varkappa - T_\sigma\varkappa\| \leq \|\varkappa - \varkappa_{n+1}\| + \|\varkappa_{n+1} - T_\sigma\varkappa\| = \|\varkappa_{n+1} - \varkappa\| + \|T_\sigma\varkappa_n - T_\sigma\varkappa\|. \tag{3.20}$$

By (3.12),

$$\|T_\sigma\varkappa_n - T_\sigma\varkappa\| \leq b\left(\|\varkappa_n - T_\sigma\varkappa\| + \|\varkappa - T_\sigma\varkappa_n\|\right). \tag{3.21}$$

Thus from (3.20),

$$\|\varkappa - T_\sigma\varkappa\| \leq (b+1)\|\varkappa_{n+1} - \varkappa\| + b\left(\|\varkappa_n - \varkappa\| + \|\varkappa - T_\sigma\varkappa\|\right), \tag{3.22}$$

which implies

$$\|\varkappa - T_\sigma\varkappa\| \leq \frac{b+1}{1-b}\|\varkappa_{n+1} - \varkappa\| + \delta\|\varkappa_n - \varkappa\|, \quad n \geq 0. \tag{3.23}$$

Letting $n \to \infty$, we get $\|\varkappa - T_\sigma\varkappa\| = 0$, i.e., $\varkappa = T_\sigma\varkappa$, so $\varkappa \in \mathrm{Fix}(T_\sigma)$. To prove uniqueness, assume $\vartheta \neq \varkappa$ is another fixed point of T_σ. By (3.12),

$$\|\varkappa - \vartheta\| \leq 2b\|\varkappa - \vartheta\|, \tag{3.24}$$

which is a contradiction since $1 \leq 2b < 1$. Therefore, $\mathrm{Fix}(T_\sigma) = \{\varkappa\}$. Given that $\mathrm{Fix}(S)$ is identical to $\mathrm{Fix}(T_\sigma)$, the unique fixed point is established. Conclusion (2) follows from (3.19).For the proof of (3), taking the limit of (3.17) and (3.18) as $m \to \infty$ provides:

$$\|\varkappa_n - \varkappa\| \leq \frac{\delta^n}{1-\delta}\|\varkappa_1 - \varkappa_0\|, \quad n \geq 1, \tag{3.25}$$

$$\|\varkappa_n - \varkappa\| \leq \frac{\delta}{1-\delta}\|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{3.26}$$

Combining these yields the unified error estimate (3.8). The scenario where $k = 0$ proceeds similarly, with $\sigma = 1$, so $T = T_1$, and the Krasnoselskij iteration (3.7) becomes Picard

iteration

$$\varkappa_{n+1} = S\varkappa_n, \quad n \geq 0. \tag{3.27}$$

This completes the proof.

□

Example 3.2.2. Given $\mathcal{X} = [0, 1]$ with its usual norm, define the mapping $S : \mathcal{X} \to \mathcal{X}$ by $S\varkappa = 1 - \varkappa$, for all $\varkappa \in [0, 1]$. Applying the properties of an averaged mapping, as outlined in definition 1.1.9, we find:

$$\| T_\sigma\varkappa - T_\sigma\vartheta\| \leq b[\|\vartheta - T_\sigma\varkappa\| + \|\varkappa - T_\sigma\vartheta\|]$$

Now, $\varkappa_{n+1} = T_\sigma(\varkappa_n), n \geq 0$ implies,

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n \tag{3.28}$$

For $\sigma = \frac{1}{4}$ initiate, $\varkappa_0 = 1$

$$\varkappa_n = \frac{1}{2} - \frac{1}{2^n}$$

So,

$$\lim_{n\to\infty} \varkappa_n = \frac{1}{2}$$

Therefore, $\frac{1}{2}$ is a unique fixed point of S.

Chapter 4

Existence and Uniqueness of Fixed Points for Enriched Ćirić-Reich-Rus Contractions in Banach and Convex Metric Settings

Among the various generalizations of classical contractions, enriched Ćirić-Reich-Rus contractions have a significant class due to their flexibility in accommodating non-linear and non-expansive mappings. Enriched Ćirić-Reich-Rus contractions is significant in proving fixed point results in Banach spaces. we see, due it's incorporation of enriched Banach and enriched Kannan contractions, it generalization of both of them. In this Chapter, we study ths definitions of Ćirić-Reich-Rus contraction and later, we will see fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces. Using Takahashi convex metric space we will extend the result to convex metric spaces. Later, we will see their generalizations can extends the results previous Chapters.

4.1 Some basic definitions

Definition 4.1.1. Ćirić [13], Reich [21], and Rus [22] each published a fixed point theorem in 1971 for mappings $S : \mathcal{X} \to \mathcal{X}$ which satisfy the ensuing condition:

$$\rho(S\varkappa, S\vartheta) \leq a\rho(\varkappa, \vartheta) + b(d(\varkappa, S\varkappa) + \rho(\vartheta, S\vartheta)), \forall \varkappa, \vartheta \in \mathcal{X} \tag{4.1}$$

where $a, b \geq 0$ and $a + 2b < 1$ is called Ćirić-Reich-Rus contractions.

Remark 4.1.1. 1. When $b = 0$, inequality 4.1 implies Banach's contraction condition

$$\rho(S\varkappa, S\vartheta) \leq a \cdot \rho(\varkappa, \vartheta), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}, \tag{4.2}$$

where $0 \leq a < 1$

2. When $a = 0$, inequality 4.1 implies Kannan's contraction condition

$$\rho(S\varkappa, S\vartheta) \leq b\big(\rho(\varkappa, S\varkappa) + \rho(\vartheta, S\vartheta)\big), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}, \tag{4.3}$$

where $0 \leq b < 1/2$.

So, results of the fixed point established in Ćirić, Reich and Rus under slightly different forms, are turning out extensions of Banach's contraction principle and Kannan's fixed point theorem as demonstrated in next example.

Example 4.1.1. [22] Consider $\mathcal{X} = [0, 1]$, $\rho(\varkappa, \vartheta) = |\varkappa - \vartheta|$, and map $f : \mathcal{X} \to \mathcal{X}$ defined as

$$f(\varkappa) = \begin{cases} \frac{7\varkappa}{20}, & \text{for } 0 \leq \varkappa \leq \frac{1}{2}, \\ \\ \frac{3\varkappa}{10}, & \text{for } \frac{1}{2} < \varkappa \leq 1. \end{cases}$$

It is evident that f is discontinuous at $\varkappa = \frac{1}{2}$, which immediately means inequality 4.2 does not hold. Furthermore, Condition 4.3 is not satisfied if we set $\varkappa = \frac{1}{2}$ and $\vartheta = 0$. In contrast, condition 4.1 can be easily shown to be satisfied by taking $a = \frac{1}{10}$ and $b = \frac{4}{9}$. Therefore, S satisfies neither Banach's contraction condition 4.2 (since S lacks continuity) nor Kannan's contraction condition 4.3. If we take $\varkappa = \frac{1}{2}$ and $\vartheta = 1$ in 4.3 , then we will get, a contradiction $\frac{1}{4} \leq 2b \cdot \frac{1}{3} < \frac{1}{3}$.

4.2 Analysis of Enriched Ćirić-Reich-Rus Contractions within Banach Spaces

Enriched Ćirić-Reich-Rus contraction is the generalization of classical Ćirić-Reich-Rus contraction. By, altering the certain conditions in the definition, we can either obtain Banach contraction or Kannan contraction and their respective generalizations.

Definition 4.2.1. [9] Given a normed linear space $(\mathcal{X}, \|\cdot\|)$, a mapping $S : \mathcal{X} \to \mathcal{X}$ is defined as a (k, b)-enriched Kannan mapping if $\exists$ constants $k \in [0, 1)$ and $b \in [0, 1/2)$ such that, for all $\varkappa, \vartheta \in \mathcal{X}$, the following inequality holds:

$$\|k(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq b\left(\|\varkappa - S\varkappa\| + \|\vartheta - S\vartheta\|\right). \tag{4.4}$$

The conventional Kannan mapping condition is obtained by setting $k = 0$ in (4.4).

Theorem 4.2.1. [9] Consider a Banach space $(\mathcal{X}, \|\cdot\|)$. If $S : \mathcal{X} \to \mathcal{X}$ is a (k, b)-enriched Kannan mapping, then:

1. There is a unique fixed point for S, specifically $\mathrm{Fix}(S) = \{\varkappa\}$ for some $\varkappa \in \mathcal{X}$. Furthermore, for a certain $\sigma \in (0, 1]$, the iterative sequence $\{\varkappa_n\}_{n=0}^{\infty}$, which is generated by

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n, \quad n \geq 0, \tag{4.5}$$

   For all $x_0 \in X$, converges to $\varkappa$;

2. The estimate that follows is accurate:

$$\|\varkappa_{n+i-1} - \varkappa\| \leq \frac{\delta^i}{1 - \delta}\|\varkappa_n - \varkappa_{n-1}\|, \quad n, i \in \mathbb{N}, \tag{4.6}$$

   where $\delta = \frac{b}{1-b}$.

In this Chapter one of the main objective is to study generalize and unify results from 1.3.1 and 4.2.1, Cconsequently, a fixed point theorem for enriched 'Ciri'c-Reich-Rus type contractions has been developed for Banach spaces.

Definition 4.2.2 ([24]). Given a normed linear space $(\mathcal{X}, \|\cdot\|)$, a mapping $S : \mathcal{X} \to \mathcal{X}$ is defined as a (k, a, b)-enriched Ćirić-Reich-Rus contraction if there exist constants $k \in [0, 1)$ and $a, b \geq 0$ satisfying $a + 2b < 1$, such that for every $\varkappa, \vartheta \in \mathcal{X}$:

$$\|k(\varkappa - \vartheta) + S\varkappa - S\vartheta\| \leq a\|\varkappa - \vartheta\| + b(\|\varkappa - S\varkappa\| + \|\vartheta - S\vartheta\|). \tag{4.7}$$

The class of Ćirić-Reich-Rus contractions is characterized by inequality (4.7) when $k = 0$.

Theorem 4.2.2. [24] Consider $(\mathcal{X}, \|\cdot\|)$ a Banach space and $S : \mathcal{X} \to \mathcal{X}$ a (k, a, b)-enriched Ćirić-Reich-Rus contraction. Then:

1. $\text{Fix}(S) = \{\varkappa\}$, for some $\varkappa \in \mathcal{X}$ and there exists $\sigma \in (0, 1]$ such that the iterative method $\{\varkappa_n\}_{n=0}^{\infty}$, given by

$$\varkappa_{n+1} = (1-\sigma)\varkappa_n + \sigma S\varkappa_n, \quad n \geq 0, \tag{4.8}$$

   For all $\varkappa_0 \in \mathcal{X}$, converges to $\varkappa$;

2. The estimate that follows is accurate:

$$\|\varkappa_{n+i-1} - \varkappa\| \leq \frac{\epsilon^i}{1-\epsilon}\|\varkappa_n - \varkappa_{n-1}\|, \quad n = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, \tag{4.9}$$

   where $\epsilon = \frac{a+b}{1-b}$.

Proof. Our initial focus is on the case where $k > 0$, noting that the scenario with $k = 0$ is trivial. Let us define the averaged mapping T_σ according to 1.1.9 with the parameter σ given by:

$$\sigma = \frac{1}{k+1} < 1. \tag{4.10}$$

From this, it follows that $k = \frac{1}{\sigma} - 1$. With this substitution, the contractive condition ?? can be re-expressed as:

$$\left\|\left(\frac{1}{\sigma} - 1\right)(\varkappa - \vartheta) + S\varkappa - S\vartheta\right\| \leq a\|\varkappa - \vartheta\| + b(\|\varkappa - S\varkappa\| + \|\vartheta - S\vartheta\|), \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{4.11}$$

which can be written equivalently as

$$\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq a\sigma\|\varkappa - \vartheta\| + b(\|\varkappa - S_\sigma\varkappa\| + \|\vartheta - S_\sigma\vartheta\|), \quad \forall \varkappa, \vartheta \in \mathcal{X}. \tag{4.12}$$

Since $a\sigma \leq a$, this implies that

$$\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq a\|\varkappa - \vartheta\| + b(\|\varkappa - S_\sigma\varkappa\| + \|\vartheta - S_\sigma\vartheta\|), \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{4.13}$$

which means that S_σ is a Ćirić-Reich-Rus contraction mapping. Using the triangle inequal-

ity in 4.35, we found out S_σ fulfils

$$\|S_\sigma \varkappa - S_\sigma \vartheta\| \leq \epsilon \|\varkappa - \vartheta\| + 2\epsilon \|\vartheta - S_\sigma \varkappa\|, \quad \forall \varkappa, \vartheta \in \mathcal{X}, \tag{4.14}$$

where $\epsilon = \frac{a+b}{1-b} < 1$. We examine the iterative sequence $\{\varkappa_n\}_{n=0}^{\infty}$ generated by 4.8. This sequence is equivalent to the Picard iteration for the mapping S_σ, which is defined as:

$$\varkappa_{n+1} = S_\sigma \varkappa_n, \quad n \geq 0. \tag{4.15}$$

Taking $\varkappa = \varkappa_{n-1}$ and $\vartheta = \varkappa_n$ in 4.36, we get

$$\|\varkappa_{n+1} - \varkappa_n\| \leq \epsilon \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{4.16}$$

Based on 4.38, the two estimates presented below are obtained:

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \epsilon^n \left(\frac{1-\epsilon^m}{1-\epsilon} \right) \|\varkappa_1 - \varkappa_0\|, \quad n \geq 0, m \geq 1, \tag{4.17}$$

and

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \epsilon \left(\frac{1-\epsilon^m}{1-\epsilon} \right) \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1, m \geq 1. \tag{4.18}$$

Given that $\{\varkappa_n\}_{n=0}^{\infty}$ is a Cauchy sequence, its convergence in the Banach space $(\mathcal{X}, \|\cdot\|)$ follows directly, according to 4.39. Let's indicate

$$\varkappa = \lim_{n \to \infty} \varkappa_n. \tag{4.19}$$

let's first show that $\varkappa$ is a fixed point of T_σ. So, we get

$$\|\varkappa - S_\sigma \varkappa\| \leq \|\varkappa - \varkappa_{n+1}\| + \|\varkappa_{n+1} - S_\sigma \varkappa\| = \|\varkappa_{n+1} - \varkappa\| + \|S_\sigma \varkappa_n - T_\sigma \varkappa\|, \tag{4.20}$$

By 4.36 we have

$$\|S_\sigma \varkappa_n - S_\sigma \varkappa\| \leq \epsilon \|\varkappa_n - \varkappa\| + 2\epsilon \|\varkappa - \varkappa_{n+1}\|, \tag{4.21}$$

and therefore from 4.41 one obtains

$$\|\varkappa - T_\sigma \varkappa\| \leq (2\epsilon + 1) \|\varkappa_{n+1} - \varkappa\| + \epsilon \|\varkappa_n - \varkappa\|, \quad n \geq 0. \tag{4.22}$$

Now, limiting $n \to \infty$ in 4.43 we will have $\|\varkappa - S_\sigma \varkappa\| = 0$, which means , $\varkappa = S_\sigma \varkappa$, so $\varkappa \in \mathrm{Fix}(S_\sigma)$. Now, for demonstrating that $\varkappa$ is the fixed point of $T_\sigma$which is unique, Similar to the derivation of 4.35, it can be shown that 4.36 holds, one can get

$$\|S_\sigma \varkappa - S_\sigma \vartheta\| \leq \epsilon \|\varkappa - \vartheta\| + 2\epsilon \|\varkappa - S_\sigma \varkappa\|, \text{ for all } \varkappa, \vartheta \in \mathcal{X}, \tag{4.23}$$

where

$$\epsilon = \frac{a+b}{1-b} < 1. \tag{4.24}$$

Assume that $\varkappa' \neq \varkappa$ is one different fixed point of S_σ. Using (4.44) and setting $\varkappa$ for the first variable and $\varkappa'$ for the second, we deduce:

$$0 < \|\varkappa - \varkappa'\| \leq \epsilon \|\varkappa - \varkappa'\| < \|\varkappa - \varkappa'\|, \tag{4.25}$$

which is a contradiction. Thus, $\mathrm{Fix}(S_\sigma) = \{\varkappa\}$. Since $\mathrm{Fix}(S) = \mathrm{Fix}(S_\sigma)$, this proves statement (1). Conclusion (2) follows by 4.19. To prove (3), we let $m \to \infty$ in 4.39 and 4.40 to get

$$\|\varkappa_n - \varkappa\| \leq \frac{\epsilon^n}{1-\epsilon} \|\varkappa_1 - \varkappa_0\|, \quad n \geq 1, \tag{4.26}$$

and

$$\|\varkappa_n - \varkappa\| \leq \frac{\epsilon}{1-\epsilon} \|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1, \tag{4.27}$$

respectively. By combining (4.47) and (4.48), we then derive the unifying error estimate 4.9.

□

Remark 4.2.1. 4.2.2 includes as particular cases 1.3.1 and 4.2.1, which are obtained from 4.2.2 for $b = 0$ and $a = 0$, respectively.

Definition 4.2.3. [24] Let $(\mathcal{X}, \|\cdot\|)$ be a linear normed space. A mapping $S : \mathcal{X} \to \mathcal{X}$ is called a generalized (k, a, b)-enriched Ćirić-Reich-Rus contraction if, for all $\varkappa, \vartheta \in \mathcal{X}$, a constant $k \in [0, 1)$ and non-negative functions $a, b : \mathcal{X} \times \mathcal{X} \to [0, 1)$ exist such that they satisfy the condition:

$$\sup_{\varkappa, \vartheta \in X} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) = \delta < 1$$

such that for all $\varkappa, \vartheta \in \mathcal{X}$, the inequality below holds:

$$\|\varkappa - \vartheta\| + \|S\varkappa - S\vartheta\| \leq a(\varkappa, \vartheta)\|\varkappa - \vartheta\| + b(\varkappa, \vartheta)\left(\|\varkappa - S\varkappa\| + \|\vartheta - S\vartheta\|\right). \tag{4.28}$$

Now, we generalizes the above theorem using similar process.

Theorem 4.2.3. [24] Given a Banach space $(\mathcal{X}, \|\cdot\|)$ and a mapping $S : \mathcal{X} \to \mathcal{X}$ that is a generalized (k, a, b)-enriched Ćirić-Reich-Rus contraction, it follows that::

1. A singleton is the fixed point set, meaning that

$$\text{Fix}(S) = \{\varkappa\}, \text{ for some } \varkappa \in \mathcal{X}. \tag{4.29}$$

2. For any initial point $\varkappa_0 \in \mathcal{X}$, there is a $\sigma \in (0, 1]$ for which the iterative method defined as

$$\varkappa_{n+1} = (1 - \sigma)\varkappa_n + \sigma S\varkappa_n, \quad n \geq 0, \tag{4.30}$$

produces a sequence $\{\varkappa_n\}_{n=0}^{\infty}$ that converges to $\varkappa$.

3. The error estimate that follows is accurate:

$$\|\varkappa_{n+i-1} - \varkappa\| \leq \frac{\delta^i}{1 - \delta}\|\varkappa_n - \varkappa_{n-1}\|, \quad n, i \in \mathbb{N} \tag{4.31}$$

where $\delta = \sup_{\varkappa, \vartheta \in \mathcal{X}} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) < 1$.

Proof. We analyze the case where $k > 0$. The result for $k = 0$ follows immediately. Then apply the averaged mapping S_σ as given in (1.1.9), where

$$\sigma = \frac{1}{k + 1} < 1. \tag{4.32}$$

This implies $k = \frac{1}{\sigma} - 1$, so the contractive condition (??) can be rewritten as

$$\left\| \left(\frac{1}{\sigma} - 1\right)(\varkappa - \vartheta) + S\varkappa - S\vartheta \right\| \leq a(\varkappa, \vartheta)\|\varkappa - \vartheta\| + b(\varkappa, \vartheta)(\|\varkappa - S\varkappa\| + \|\vartheta - S\vartheta\|), \tag{4.33}$$

for all $\varkappa, \vartheta \in \mathcal{X}$. This is equivalent to

$$\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq a(\varkappa, \vartheta)\sigma\|\varkappa - \vartheta\| + b(\varkappa, \vartheta)(\|\varkappa - S_\sigma\varkappa\| + \|\vartheta - S_\sigma\vartheta\|). \tag{4.34}$$

Since $a(\varkappa,\vartheta)\sigma \leq a(\varkappa,\vartheta)$, it follows that

$$\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq a(\varkappa,\vartheta)\|\varkappa-\vartheta\| + b(\varkappa,\vartheta)(\|\varkappa - S_\sigma\varkappa\| + \|\vartheta - S_\sigma\vartheta\|), \tag{4.35}$$

which shows that S_σ is a Ćirić-Reich-Rus type contraction. Applying the triangle inequality and taking the supremum on both sides of (4.35), we deduce

$$\|S_\sigma\varkappa - S_\sigma\vartheta\| \leq \delta\|\varkappa-\vartheta\| + 2\delta\|\vartheta - S_\sigma\varkappa\|, \tag{4.36}$$

for all $\varkappa,\vartheta \in \mathcal{X}$, where

$$\delta = \sup_{\varkappa,\vartheta\in\mathcal{X}} (a(\varkappa,\vartheta) + 2b(\varkappa,\vartheta)) < 1. \tag{4.37}$$

Define a sequence $\{\varkappa_n\}_{n=0}^{\infty}$ via the Picard iteration: $\varkappa_{n+1} = S_\sigma\varkappa_n, \quad n \geq 0$. Taking $\varkappa = \varkappa_{n-1}$ and $\vartheta = \varkappa_n$ in (4.36), we obtain

$$\|\varkappa_{n+1} - \varkappa_n\| \leq \delta\|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{4.38}$$

From this, the following estimates follow:

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \delta^n \left(\frac{1-\delta^m}{1-\delta}\right) \|\varkappa_1 - \varkappa_0\|, \quad n \geq 0, m \geq 1, \tag{4.39}$$

and

$$\|\varkappa_{n+m} - \varkappa_n\| \leq \delta \left(\frac{1-\delta^m}{1-\delta}\right) \|\varkappa_n - \varkappa_{n-1}\|, \quad n, m \geq 1. \tag{4.40}$$

As $\{\varkappa_n\}$ is a Cauchy sequence in the Banach space $(\mathcal{X}, \|\cdot\|)$, it is therefore convergent. Let $\varkappa = \lim_{n\to\infty} \varkappa_n$ be its limit. We now proceed to show that $\varkappa$ is a fixed point of S_σ. From the triangle inequality, it follows:

$$\|\varkappa - S_\sigma\varkappa\| \leq \|\varkappa - \varkappa_{n+1}\| + \|\varkappa_{n+1} - S_\sigma\varkappa\| = \|\varkappa_{n+1} - \varkappa\| + \|S_\sigma\varkappa_n - S_\sigma\varkappa\|. \tag{4.41}$$

Using (4.36), we get

$$\|S_\sigma\varkappa_n - S_\sigma\varkappa\| \leq \delta\|\varkappa_n - \varkappa\| + 2\delta\|\varkappa - \varkappa_{n+1}\|, \tag{4.42}$$

and thus from (4.41),

$$\|\varkappa - S_\sigma \varkappa\| \leq (2\delta + 1)\|\varkappa_{n+1} - \varkappa\| + \delta\|\varkappa_n - \varkappa\|, \quad n \geq 0. \tag{4.43}$$

Taking the limit as $n \to \infty$ yields $\|\varkappa - S_\sigma \varkappa\| = 0$, hence $\varkappa = S_\sigma \varkappa$, i.e., $\varkappa \in \text{Fix}(S_\sigma)$. To establish uniqueness, we note that (4.35) implies a form of inequality analogous to (4.36), namely:

$$\|S_\sigma \varkappa - S_\sigma \vartheta\| \leq \delta\|\varkappa - \vartheta\| + 2\delta\|\varkappa - S_\sigma \varkappa\|, \tag{4.44}$$

for all $\varkappa, \vartheta \in \mathcal{X}$, with

$$\epsilon = \sup_{\varkappa, \vartheta \in \mathcal{X}} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) < 1. \tag{4.45}$$

Suppose $\varkappa \neq \varkappa'$ are both fixed points of S_σ. Then (4.44) with $\varkappa = \varkappa$ and $\vartheta = \varkappa'$ gives

$$0 < \|\varkappa - \varkappa'\| \leq \delta\|\varkappa - \varkappa'\| < \|\varkappa - \varkappa'\|, \tag{4.46}$$

which is a contradiction. Therefore, $\text{Fix}(S_\sigma) = \{\varkappa\}$, and since $\text{Fix}(S) = \text{Fix}(S_\sigma)$, Claim (1) is proven. Claim (2) follows from the convergence in (4.19). To establish (3), letting $m \to \infty$ in (4.39) and (4.40), we get

$$\|\varkappa_n - \varkappa\| \leq \frac{\delta^n}{1 - \delta}\|\varkappa_1 - \varkappa_0\|, \quad n \geq 1, \tag{4.47}$$

and

$$\|\varkappa_n - \varkappa\| \leq \frac{\delta}{1 - \delta}\|\varkappa_n - \varkappa_{n-1}\|, \quad n \geq 1. \tag{4.48}$$

Combining (4.47) and (4.48), we obtain the unified error estimate (4.9).

□

4.3 Analysis of Enriched Ćirić-Reich-Rus Contractions in Convex Metric Spaces

In 1970, Takahashi [23] introduced a generalized convexity framework outside the realm of linear spaces, which has proven to be highly effective in the study of fixed point approximation.

Definition 4.3.1. [23] Given a metric space $(\mathcal{X}, \rho)$, a function $W : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ is referred to as a convex structure on $\mathcal{X}$ if it is continuous and, for any $\varkappa, \vartheta \in \mathcal{X}$ and $\sigma \in [0,1]$, the subsequent inequality is observed:

$$\rho(u, W(\varkappa, \vartheta, \sigma)) \leq \sigma \cdot \rho(u, \varkappa) + (1-\sigma) \cdot \rho(u, \vartheta), \quad \forall u \in \mathcal{X}, \tag{4.49}$$

holds. A metric space furnished with such a convex structure W is termed a Takahashi convex metric space and is denoted as $(\mathcal{X}, \rho, W)$.

Remark 4.3.1. Clearly, every linear normed space, along with each of its convex subsets, forms a convex metric space endowed with the natural convex structure

$$W(\varkappa, \vartheta, \sigma) = \sigma\varkappa + (1-\sigma)\vartheta, \quad \varkappa, \vartheta \in \mathcal{X}, \quad \sigma \in [0,1], \tag{4.50}$$

however, the converse does not always hold: not all convex metric spaces can be isometrically embedded into a Banach space; indeed, some examples exist that preclude such an embedding.

Example 4.3.1. Let $\mathcal{X} = \{(\varkappa_1, \varkappa_2) \in \mathbb{R}^2 : \varkappa_1, \varkappa_2 > 0\}$. For $\varkappa = (\varkappa_1, \varkappa_2)$, $\vartheta = (\vartheta_1, \vartheta_2)$ in $\mathcal{X}$ and $\alpha \in [0,1]$, consider a map defined as $W : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ by

$$W(\varkappa, \vartheta; \alpha) = \left(\alpha\varkappa_1 + (1-\alpha)\vartheta_1, \frac{\alpha\varkappa_1\varkappa_2 + (1-\alpha)\vartheta_1\vartheta_2}{\alpha\varkappa_1 + (1-\alpha)\vartheta_1} \right).$$

Also, define a metric $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\rho(\varkappa, \vartheta) = |\varkappa_1 - \vartheta_1| + |\varkappa_1\varkappa_2 - \vartheta_1\vartheta_2|.$$

The fact that $\mathcal{X}$ is a convex metric space rather than a normed space can be confirmed.

Lemma 4.3.1. [23] [10] [1] Consider $(\mathcal{X}, \rho, W)$ a convex metric space. For each $\varkappa, \vartheta \in \mathcal{X}$ and $\sigma, \sigma_1, \sigma_2 \in [0,1]$, satisfies the following conditions:

1. $\rho(\varkappa, \vartheta) = \rho(\varkappa, W(\varkappa, \vartheta; \sigma)) + \rho(W(\varkappa, \vartheta; \sigma), \vartheta)$.
2. $\rho(\varkappa, W(\varkappa, \vartheta; \sigma)) = (1-\sigma)\rho(\varkappa, \vartheta)$ and $\rho(W(\varkappa, \vartheta; \sigma), \vartheta) = \sigma\rho(\varkappa, \vartheta)$.
3. $W(\varkappa, \varkappa; \sigma) = \varkappa$, $W(\varkappa, \vartheta; 0) = \vartheta$, and $W(\varkappa, \vartheta; 1) = \varkappa$.

4. $|\sigma_1 - \sigma_2|\rho(\varkappa, \vartheta) \leq \rho(W(\varkappa, \vartheta; \sigma_1), W(\varkappa, \vartheta; \sigma_2))$.

Lemma 4.3.2. [1] In a convex metric space $(\mathcal{X}, \rho, W)$, consider a self-mapping $S : \mathcal{X} \to \mathcal{X}$. The mapping $S_\sigma : \mathcal{X} \to \mathcal{X}$ is then defined as:

$$S_\sigma \varkappa = W(\varkappa, S\varkappa, \sigma), \quad \forall \varkappa \in \mathcal{X}. \tag{4.51}$$

Then, for every $\sigma \in [0, 1]$,

$$\mathrm{Fix}(S) = \mathrm{Fix}(S_\sigma). \tag{4.52}$$

Definition 4.3.2. [24] A mapping $S : \mathcal{X} \to \mathcal{X}$ operating within a convex metric space $(\mathcal{X}, \rho, W)$ is designated a (k, a, b)-enriched Ćirić-Reich-Rus contraction if there are constants $k \in [0, 1)$ and $a, b \geq 0$ such that $a + 2b < 1$, and for any $\varkappa, \vartheta \in \mathcal{X}$:,

$$\rho\big(W(\varkappa, S\varkappa, \sigma), W(\vartheta, S\vartheta, \sigma)\big) \leq a\,\rho(\varkappa, \vartheta) + b\left(\rho(\varkappa, W(\varkappa, S\varkappa, \sigma)) + \rho(\vartheta, W(\vartheta, S\vartheta, \sigma))\right). \tag{4.53}$$

Example 4.3.2. [24] Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, and consider the convex structure $W : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ given by

$$W(\varkappa, \vartheta, \sigma) = \sigma\varkappa + (1 - \sigma)\vartheta, \quad \forall \varkappa, \vartheta \in \mathcal{X},\ \sigma \in [0, 1]. \tag{4.54}$$

Under this setting, any (k, a, b)-enriched Ćirić-Reich-Rus contraction $S : \mathcal{X} \to \mathcal{X}$ as defined in 4.3.2 also satisfies the conditions of a (k, a, b)-enriched Ćirić-Reich-Rus contraction according to Definition ??.

Theorem 4.3.1. [24] Assume $(\mathcal{X}, \rho, W)$ is a convex metric space which is complete, and $S : \mathcal{X} \to \mathcal{X}$ is a (k, a, b)-enriched Ćirić-Reich-Rus contraction. We then have:

1. S has a unique fixed point i.e.,

$$\mathrm{Fix}(S) = \{\mathfrak{p}\},\ \text{for some}\ \mathfrak{p} \in \mathcal{X}. \tag{4.55}$$

2. The iterative process yields the sequence $\{\varkappa_n\}_{n=0}^{\infty}$

$$\varkappa_{n+1} = W(\varkappa_n, S\varkappa_n, \sigma), \quad n \geq 0, \tag{4.56}$$

converges to $\mathfrak{p}$, for any initial point $x_0 \in \mathcal{X}$.

3. The subsequent error estimate is valid:

$$\rho(\varkappa_{n+i-1}, \mathfrak{p}) \leq \frac{\epsilon^i}{1-\epsilon}\rho(\varkappa_n, \varkappa_{n-1}), \quad n, i \in \mathbb{N} \tag{4.57}$$

where

$$\epsilon = \frac{a+b}{1-b} < 1. \tag{4.58}$$

Proof. The mapping $S_\sigma : \mathcal{X} \to \mathcal{X}$ defined by the enriched contractive condition 4.53

$$S_\sigma \varkappa = W(\varkappa, S\varkappa, \sigma), \quad \varkappa \in \mathcal{X} \tag{4.59}$$

satisfies

$$\rho(S_\sigma \varkappa, S_\sigma \vartheta) \leq a \cdot \rho(\varkappa, \vartheta) + b \cdot (\rho(\varkappa, S_\sigma \varkappa) + \rho(\vartheta, S_\sigma \vartheta)), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}. \tag{4.60}$$

This demonstrates that S_σ is a Ćirić-Reich-Rus contraction. The corresponding Picard iteration for S_σ is the Krasnoselskij iterative process $\{\varkappa_n\}_{n=0}^{\infty}$, defined as:

$$\varkappa_{n+1} = S_\sigma \varkappa_n, \quad n \geq 0. \tag{4.61}$$

From the triangle inequality applied to (4.80), we obtain

$$\rho(S_\sigma \varkappa, S_\sigma \vartheta) \leq \epsilon \cdot \rho(\varkappa, \vartheta) + 2\epsilon \cdot \rho(\vartheta, S_\sigma \varkappa), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}, \tag{4.62}$$

where

$$\epsilon = \frac{a+b}{1-b} < 1. \tag{4.63}$$

Now, by setting $\varkappa = \varkappa_n$ and $\vartheta = \varkappa_{n-1}$ in (4.82), we get

$$\rho(\varkappa_{n+1}, \varkappa_n) \leq \epsilon \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1. \tag{4.64}$$

This inequality implies, by induction,

$$\rho(\varkappa_{n+1}, \varkappa_n) \leq \epsilon^n \cdot \rho(x_1, x_0), \quad n \geq 1. \tag{4.65}$$

From this, we can derive two important estimates:

$$\rho(\varkappa_{n+m}, \varkappa_n) \leq \frac{\epsilon^n(1-\epsilon^m)}{1-\epsilon} \cdot \rho(\varkappa_1, \varkappa_0), \quad n \geq 0,\, m \geq 1, \tag{4.66}$$

and

$$\rho(\varkappa_{n+m}, \varkappa_n) \leq \frac{\epsilon(1-\epsilon^m)}{1-\epsilon} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1,\, m \geq 1. \tag{4.67}$$

The sequence $\{\varkappa_n\}_{n=0}^{\infty}$ is Cauchy and, as a result, convergent in the entire convex metric space $(\mathcal{X}, \rho)$, according to (4.86). Let us

$$\mathfrak{p} = \lim_{n\to\infty} \varkappa_n. \tag{4.68}$$

The fixed point of T_σ, $\mathfrak{p}$, is then confirmed to exist. We have

$$\rho(\mathfrak{p}, S_\sigma \varkappa) \leq \rho(\mathfrak{p}, \varkappa_{n+1}) + \rho(\varkappa_{n+1}, S_\sigma \mathfrak{p}) \leq (2\epsilon+1)\rho(\varkappa_{n+1}, \mathfrak{p}) + \epsilon\rho(\varkappa_n, \mathfrak{p}), \tag{4.69}$$

and letting $n \to \infty$, this yields $\rho(\mathfrak{p}, S_\sigma \mathfrak{p}) = 0$, which means $\mathfrak{p} = S_\sigma \mathfrak{p}$, i.e., $\mathfrak{p} \in \mathrm{Fix}(S_\sigma)$. To demonstrate that $\mathfrak{p}$ is the fixed point of S_σ which is unique, assuming $\vartheta \neq \varkappa$ is one other fixed point different from $\mathfrak{p}$. Then, from (4.82) with $\varkappa = \mathfrak{p}$ and $\vartheta = \mathfrak{q}$,

$$0 < \rho(\mathfrak{p}, \mathfrak{q}) \leq \epsilon \cdot \rho(\mathfrak{p}, \mathfrak{q}) < \rho(\mathfrak{p}, \mathfrak{q}), \tag{4.70}$$

which is a contradiction. Therefore, $\mathrm{Fix}(S_\sigma) = \{\mathfrak{p}\}$. Given that $\mathrm{Fix}(S) = \mathrm{Fix}(S_\sigma)$, this establishes part (1) of the theorem. For part (2), the convergence follows from (4.88). For part (3), letting $m \to \infty$ in (4.86) and (4.87) gives

$$\rho(\varkappa_n, \mathfrak{p}) \leq \frac{\epsilon^n}{1-\epsilon} \cdot \rho(\varkappa_1, \varkappa_0), \quad n \geq 1, \tag{4.71}$$

and

$$\rho(\varkappa_n, \mathfrak{p}) \leq \frac{\epsilon}{1-\epsilon} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1. \tag{4.72}$$

Combining these estimates yields the unified error bound

$$\rho(\varkappa_{n+i-1}, \mathfrak{p}) \leq \frac{\epsilon^i}{1-\epsilon} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1,\ i \geq 1, \tag{4.73}$$

which completes the proof. □

Definition 4.3.3. [24] In a convex metric space $(\mathcal{X}, \rho, W)$, a mapping $S : \mathcal{X} \to \mathcal{X}$ is called a generalized (k, a, b)-enriched Ćirić-Reich-Rus contraction if, $\forall \varkappa, \vartheta \in \mathcal{X}$, we can find a constant $k \in [0, 1)$. Also with non-negative functions $a, b : \mathcal{X} \times \mathcal{X} \to [0, 1)$ which meet the following condition:

$$\sup_{\varkappa, \vartheta \in X} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) = \delta < 1 \tag{4.74}$$

such that $\forall \varkappa, \vartheta \in X$,

$$\rho(W(\varkappa, S\varkappa, \sigma), W(\vartheta, S\vartheta, \sigma)) \leq a(\varkappa, \vartheta)\rho(\varkappa, \vartheta) + \rho(\varkappa, \vartheta) \left(\rho(\varkappa, W(\varkappa, S\varkappa, \sigma)) + \rho(\vartheta, W(\vartheta, S\vartheta, \sigma))\right). \tag{4.75}$$

Remark 4.3.2. Evidently, any (k, a, b)-enriched Ćirić-Reich-Rus contraction also qualifies as a generalized (k, a, b)-enriched Ćirić-Reich-Rus contraction. Nevertheless, the reverse is not true in general.

Theorem 4.3.2. [24] Given a convex metric space $(\mathcal{X}, \rho, W)$ which is complete and a mapping $S : \mathcal{X} \to \mathcal{X}$ that is a generalized (k, a, b)-enriched Ćirić-Reich-Rus contraction, it follows that::

1. The fixed point set of T is a singleton:

$$\text{Fix}(S) = \{\varkappa\}, \text{ for some } \varkappa \in \mathcal{X}. \tag{4.76}$$

2. The iterative procedure yields the sequence $\{\varkappa_n\}_{n=0}^{\infty}$

$$\varkappa_{n+1} = W(\varkappa_n, S\varkappa_n, \sigma), \quad n \geq 0, \tag{4.77}$$

For every initial point $\varkappa_0 \in \mathcal{X}$, it converges to $\varkappa$.

3. The error estimate that follows is accurate:

$$\rho(\varkappa_{n+i-1}, \varkappa) \leq \frac{\delta^i}{1-\delta} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n, i \in \mathbb{N} \tag{4.78}$$

where $\delta = \sup_{\varkappa, \vartheta \in X} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) < 1$.

Proof. The mapping $S_\sigma : \mathcal{X} \to \mathcal{X}$ defined by the enriched contractive condition 4.53

$$S_\sigma \varkappa = W(\varkappa, S\varkappa, \sigma), \quad \varkappa \in \mathcal{X} \tag{4.79}$$

satisfies

$$\rho(S_\sigma \varkappa, S_\sigma \vartheta) \leq a(\varkappa, \vartheta) \cdot \rho(\varkappa, \vartheta) + b(\varkappa, \vartheta) \cdot (\rho(\varkappa, S_\sigma \varkappa) + \rho(\vartheta, S_\sigma \vartheta)), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}. \tag{4.80}$$

This demonstrates that S_σ is a Ćirić-Reich-Rus contraction. The corresponding Picard iteration for S_σ is the Krasnoselskij iterative process $\{\varkappa_n\}_{n=0}^{\infty}$, which is defined as:

$$\varkappa_{n+1} = S_\sigma \varkappa_n, \quad n \geq 0. \tag{4.81}$$

From the triangle inequality and taking supremum on both sides of equation (4.80), we obtain

$$\rho(S_\sigma \varkappa, S_\sigma \vartheta) \leq \delta \cdot \rho(\varkappa, \vartheta) + 2\delta \cdot \rho(\vartheta, S_\sigma \varkappa), \quad \text{for all } \varkappa, \vartheta \in \mathcal{X}, \tag{4.82}$$

where

$$\delta = \sup_{\varkappa, \vartheta \in X} (a(\varkappa, \vartheta) + 2b(\varkappa, \vartheta)) < 1. \tag{4.83}$$

Now, by setting $\varkappa = \varkappa_n$ and $\vartheta = \varkappa_{n-1}$ in (4.82), we get

$$\rho(\varkappa_{n+1}, \varkappa_n) \leq \delta \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1. \tag{4.84}$$

This inequality implies, by induction,

$$\rho(\varkappa_{n+1}, \varkappa_n) \leq \delta^n \cdot \rho(\varkappa_1, \varkappa_0), \quad n \geq 1. \tag{4.85}$$

From this, we can derive two important estimates:

$$\rho(\varkappa_{n+m}, \varkappa_n) \leq \frac{\delta^n(1-\delta^m)}{1-\delta} \cdot \rho(\varkappa_1, \varkappa_0), \quad n \geq 0,\, m \geq 1, \tag{4.86}$$

and

$$\rho(\varkappa_{n+m}, \varkappa_n) \leq \frac{\delta(1-\delta^m)}{1-\delta} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1,\, m \geq 1. \tag{4.87}$$

From (4.86), it follows that the sequence $\{\varkappa_n\}_{n=0}^{\infty}$ is Cauchy. So, it converges within the complete convex metric space $(\mathcal{X}, \rho)$. Assume

$$\varkappa = \lim_{n\to\infty} \varkappa_n. \tag{4.88}$$

Next, we verify that $\varkappa$ is a fixed point of S_σ. It follows that:

$$\rho(\varkappa, S_\sigma\varkappa) \leq \rho(\varkappa, \varkappa_{n+1}) + \rho(\varkappa_{n+1}, S_\sigma\varkappa) \leq (2\epsilon+1)\rho(\varkappa_{n+1}, \varkappa) + \epsilon\rho(\varkappa_n, \varkappa), \tag{4.89}$$

and letting $n \to \infty$, this yields

$$\rho(\varkappa, S_\sigma\varkappa) = 0, \tag{4.90}$$

which means $\varkappa = S_\sigma\varkappa$, i.e., $\varkappa \in \mathrm{Fix}(S_\sigma)$. To demonstrate that $\varkappa$ is the fixed point of S_σ which is unique, assume that $\varkappa \neq \varkappa'$ is one other fixed point different from $\varkappa$. Then, from (4.82) with $\varkappa = \varkappa$ and $\vartheta = \varkappa'$,

$$0 < \rho(\varkappa, \varkappa') \leq \epsilon \cdot \rho(\varkappa, \varkappa') < \rho(\varkappa, \varkappa'), \tag{4.91}$$

which is a contradiction.This means $\mathrm{Fix}(S_\sigma) = \{\varkappa\}$. Given that $\mathrm{Fix}(S) = \mathrm{Fix}(S_\sigma)$, this establishes part (1) of the theorem. For part (2), the convergence follows from (4.88). For part (3), letting $m \to \infty$ in (4.86) and (4.87) gives

$$\rho(\varkappa_n, \varkappa) \leq \frac{\delta^n}{1-\delta} \cdot \rho(\varkappa_1, \varkappa_0), \quad n \geq 1, \tag{4.92}$$

and

$$\rho(\varkappa_n, \varkappa) \leq \frac{\delta}{1-\delta} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1. \tag{4.93}$$

Combining these estimates yields the unified error bound

$$\rho(\varkappa_{n+i-1}, \varkappa) \leq \frac{\delta^i}{1-\delta} \cdot \rho(\varkappa_n, \varkappa_{n-1}), \quad n \geq 1,\, i \geq 1, \tag{4.94}$$

which completes the proof. □

Chapter 5

Conclusion

Conclusion

1. Study of different kinds of enriched fixed point theorems provides more in depth analysis of fixed points in abstract spaces and also generalizes more primitive fixed point theorems like, Banach fixed point theorem.

2. The study suggests that strong and weak convergence study of fixed points of enriched contractions depending upon the type of contraction mappings used in abstract spaces such as contractive mapping or nonexpansive mapping.

3. Study suggests that to approximate fixed point more precisely in Banach spaces and in convex metric setting, we have more elaborative and generalize definitions of contraction mappings . Later, we found that to approximate fixed point in enriched contraction. We do not need rely on kransnoselskij iterative scheme, instead we could use Takahashi's convex structure to generalize the iterative schemes.

References

[1] R. P. Agarwal, Donal O'Regan, and D. R. Sahu. Fixed point theory for Lipschitzian-type mappings with applications 6, New York: Springer, 2009.

[2] R. Anjum, and Mujahid Abbas, "Common fixed point theorem for modified Kannan enriched contraction pair in Banach spaces and its applications," Filomat 35.8, 2485-2495, (2021).

[3] Baillon, Jean-Bernard, "On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces"Houston Journal of Mathematics 4, 1-9, (1978).

[4] Borwein, Jonathan, Simeon Reich, and Itai Shafrir "Krasnoselski-Mann iterations in normed spaces", Canadian Mathematical Bulletin 35.1, 21-28, (1992).

[5] V. Berinde, and F. Takens, Iterative approximation of fixed points Vol. 1912, Berlin: Springer, (2007).

[6] V. Berinde, "Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces," Carpathian Journal of Mathematics 35.3, 293-304 (2019).

[7] V. Berinde, and Mădălina Păcurar, "Approximating fixed points of enriched contractions in Banach spaces," Journal of Fixed Point Theory and Applications 22, 1-10 (2020).

[8] V. Berinde, and Mădălina Păcurar, "Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces," Journal of Fixed Point Theory and Applications 23, 1-16 (2021).

[9] V. Berinde and Mădălina Păcurar,"Kannan's fixed point approximation for solving split feasibility and variational inequality problems", Journal of Computational and Applied Mathematics 386, 113217, (2021).

[10] Felix E. Browder, and Wolodymyr V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," Journal of Mathematical Analysis and Applications 20.2, 197-228 (1967).

[11] R. Caccioppoli, "Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale," Rend. Accad. Naz. Lincei 11, 794-799 (1930).

[12] S. K. Chatterjea, "Fixed-point theorems," Comptes Rendus de l'Académie Bulgare des Sciences 25, pp. 15-18, (1972).

[13] L. B. Ciric, "A generalization of Banach's contraction principle," Proceedings of the American Mathematical Society 45, 267-273 (1974).

[14] E. H. Connell, "Properties of fixed point spaces," Proceeding of American Mathematical Society 10, 974-979 (1959).

[15] R. Kannan , "Some results on fixed points," Bulletin of the Calcutta Mathematical Society 60, 71-76 (1968).

[16] R. Kannan, "Some results on fixed points—II," The American Mathematical Monthly 76.4, 405-408 (1969).

[17] E. Kreyszig, Introductory functional analysis with applications, 17, John Wiley and Sons, (1991).

[18] Krasnoselśkii, Mark Aleksandrovich. "Two remarks on the method of successive approximations". Uspekhi matematicheskikh nauk 10.1, 123-127, (1955).

[19] A. R. Lovaglia , "Locally uniformly convex Banach spaces," Transactions of the American Mathematical Society 78.1, 225-238, (1955).

[20] W. V. Petryshyn, "Construction of fixed points of demicompact mappings in Hilbert space," Journal of Mathematical Analysis and Applications 14.2, 276-284 (1966).

[21] S. Reich, "Some remarks concerning contraction mappings," Canadian Mathematical Bulletin 14, 121-124 (1971).

[22] I. A. Rus, "Some fixed point theorems in metric spaces," Rendiconti dell'Istituto di Matematica dell'Università di Trieste, 3, 169-172 (1971)(1972).

[23] T. Wataru, "A convexity in metric space and nonexpansive mappings I," Kodai Mathematical Seminar Reports. Vol. 22. No. 2. Department of Mathematics, Tokyo Institute of Technology, 1970.

[24] V. Berinde, and Mădălina Păcurar, "Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces," Carpathian Journal of Mathematics 37.2, 173-184 (2021).